The Wilson's Meet And Reciprocal Wilson's Meet Matrices On A Sets

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Abstract

Wilson's meet and Reciprocal Wilson's meet matrices on A sets are considered and the expression for determinant, inverse and the matrices are expressed in terms of A-sets.

Key words : Meet Matrices, Wilson's Meet Matrices, Reciprocal Wilson's Meet Matrices, a- Set, A-Set

1 Introduction

Let $S = \{x_1, x_2, ..., x_n\}$ be a set of distinct positive integesr ,and let f be an arithmetical function. Then $n \times n$ matrix (S) whose i,j-entry is the greatest common divisor (x_i, x_j) of x_i and x_j is called the GCD matrix on S[1, 2, 4, 5].The set S is said to be *factor-closed* if it contains every divisor of any element of S,and the set S is said to be *GCD-closed* if it contains the greatest common divisor of any two elements of S[2, 3, 7].
Let $(P, \wedge)$ be a meet-semilattice and let $S = \{x_1, x_2, \ldots\ldots\ldots\ldots x_n\}$ be a subset of P. Then S is an A-set if $A = \{x_i \wedge x_j \ / \ x_i \neq x_j\}$ is a chain. For example, chains and a-sets (with $A = \{a\}$) are known trivial A-sets. The meet matrix $(S)_f$ on S with respect to a function $f : P \rightarrow \mathbf{C}$ is defined as
$((S)_f)_{ij} = f(x_i \wedge x_j)$.

If $f(x_i \wedge x_j) = (x_i \wedge x_j - 1)! + 1$ then the n × n meet matrix obtained is called the Wilson's meet matrix on S.

If $f(x_i \wedge x_j) = \frac{1}{(x_i \wedge x_j - 1)! + 1}$ then the n × n meet matrix obtained is called the Reciprocal Wilson's meet matrix on S.

We say that S is an ***A*-set** if the set $A = \{x_i \wedge x_j / x_i \neq x_j\}$ is a chain (an A-set need not be meet-closed). For example, chains and a-sets (with $A = \{a\}$ are known trivial A sets.
Since the method, presented in [10], adapted to A-sets might not be sufficiently effective, we give a new structure theorem for $(S)_f$ where S is an A-set. One of its features is that it supports recursive function calls.
By the structure theorem we obtain a recursive formula for $\det(S)_f$ and for $(S_f)^{-1}$ on A-sets. By dissolving the recursion on certain sets we also obtain e.g. the known explicit determinant and inverse formulae on chains and a-sets.
Note that $(\mathbf{Z}+,|) = (\mathbf{Z}+, \gcd, \text{lcm})$ is a locally finite lattice, where | is the usual divisibility relation and gcd and lcm stand for the greatest common divisor and the least common multiple of integers. Thus meet matrices are generalizations of GCD matrices $((S)_f)_{ij} = f(\gcd(x_i, x_j))$ and therefore the results in this paper also hold for GCD. For general accounts of GCD matrices, see [11]. Meet matrices are also generalizations of GCUD matrices, the unitary analogies of GCD matrices, see [12]. Thus the results also hold for GCUD matrices.

2 DEFINITIONS

Definition 2.1 *The binary operation* $\sqcap$ *is defined by*

$$S_1 \sqcap S_2 = [x \wedge y \,/\, x \in S_1, y \in S_2, x \neq y\} \qquad (2.1)$$

where S_1 *and* S_2 *are nonempty subsets of* P. *Let* S *be a subset of* P *and let* $a \in P$. *If* $S \sqcap S = \{a\}$, *then the set* S *is said to be an* ***a-set***.

Definition 2.2 *Let* $S = \{ x_1, x_2, \ldots\ldots\ldots\ldots x_n \}$ *be a subset of* P *with* $x_i < x_j \Rightarrow i < j$ *and let* $A = \{ a_1, a_2, \ldots\ldots\ldots\ldots a_{n-1} \}$ *be a multichain (i.e. a chain where duplicates are allowed) with* $a_1 \leq a_2 \leq \ldots\ldots\ldots\ldots \leq a_{n-1}$. *The set* S *is said to be an* ***A-set*** *if* $\{x_k\} \sqcap \{x_{k+1}, \ldots\ldots, x_n\} = \{a_k\}$ *for all* $k = 1, 2, \ldots\ldots, n$-1.
Every chain $S = \{ x_1, x_2, \ldots\ldots\ldots\ldots x_n \}$ is an A-set with $A = S \setminus \{x_n\}$ and every a-set is always an A-set with $A = \{a\}$.

Definition 2.3 *Let* f *be a complex-valued function on* P. *Then the* $n \times n$ *matrix* $(S)_f$, *where* $((S)_f)_{ij} = f(x_i \wedge x_j)$, *is called the meet matrix on* S *with respect to* f. *Also the* $n \times n$ *matrix* $(S)_f$, *where* $((S)_f)_{ij} = f(x_i \wedge x_j) = (x_i \wedge x_j - 1)! + 1$ is called the Wilson's meet matrix on S.

Definition 2.4 *Let* f *be a complex-valued function on* P. *Also the* $n \times n$ *matrix* $(S)_f$, *where* $((S)_f)_{ij} = f(x_i \wedge x_j) = \frac{1}{(x_i \wedge x_j - 1)! + 1}$ is called the Reciprocal Wilson's meet matrix on S.
In what follows, let $S = \{ x_1, x_2, \ldots\ldots\ldots\ldots x_n \}$ always be a finite subset of P with $x_i < x_j \Rightarrow i < j$. Let also $A = \{ a_1, a_2, \ldots\ldots\ldots\ldots a_{n-1} \}$ with $a_i < a_j \Rightarrow i < j$. Note that S has always n distinct elements, but it is possible that the set A is a multiset. Let f be a complex-valued function on P.

3 WILSON'S AND RECIPROCAL WILSON'S MEET MATRICES ON *A*-SETS

3.1 Structure Theorem

Theorem 3.1 (Structure Theorem) *Let* $S = \{ x_1, x_2, \ldots\ldots\ldots\ldots x_n \}$ *be an* A*-set, where* $A = \{ a_1, a_2, \ldots\ldots\ldots\ldots a_{n-1} \}$ *is a multichain. Let* $f_1, f_2, \ldots\ldots, f_n$ *denote the functions on* P *defined by* $f_1 = f$ *and*

$$f_{k+1}(x) = f_k(x) - \frac{f_k(a_k)^2}{f_k(x_k)} \qquad (3.1)$$

for $k = 1, 2, \ldots\ldots\ldots, n - 1$.
Then

$$(S)_f = M^T D M, \qquad (3.2)$$

where $D = \mathrm{diag}\,(f_1(x_1), f_2(x_2), \ldots\ldots\ldots, f_n(x_n))$ *and* M *is the* $n \times n$ *upper triangular matrix with* 1*'s on its main diagonal, and further*

$$(M)_{ij} = \frac{f_i(a_i)}{f_i(x_i)} \qquad (3.3)$$

for all $i < j$. *(Note that* $f_1, \ldots\ldots\ldots, f_n$ *exist if and only if* $(f_k(x_k) = 0, a_k \neq x_k) \Rightarrow f_k(a_k) = 0$ *holds for all* $k = 1, 2, \ldots\ldots\ldots, n$-1. *In the case* $f_k(a_k) = f_k(x_k) = 0$ *we can write e.g.* $(M)_{kj} = 0$ *for all* $k < j$.
Proof : Let $i < j$. Then

$(M^TDM)_{ij} = \sum_{k=1}^{n} (M)_{ki}(D)_{kk}(M)_{kj} = f_i(a_i) + \sum_{k=1}^{i-1} \frac{f_k(a_k)^2}{f_k(x_k)}$ (3.4)

$= f_i(a_i) + \sum_{k=1}^{i-1} (f_k(a_i) - f_{k+1}(a_i)) = f_1(a_i) = f(x_i \wedge x_j)$.

The case $i = j$ is similar, we only replace every a_i with x_i in (3.4). Since M^TDM is symmetric, we do not need to treat the case $i > j$.

3.2 Determinant of Wilson's Meet and Reciprocal Wilson's Meet matrices on *A*-sets

By Structure Theorem we obtain a new recursive formula for $\det(S)_f$ on *A*-sets.

Theorem 3.2 *Let* $S = \{ x_1, x_2, \ldots\ldots\ldots\ldots x_n \}$ *be an A-set, where* $A = \{ a_1, a_2, \ldots\ldots\ldots\ldots a_{n-1} \}$

is a multichain. Let $f_1, f_2, \ldots\ldots, f_n$ *be the functions defined in* (3.1). *Then*

$$\det (S)_f = f_1(x_1)f_2(x_2)\ldots\ldots\ldots f_n(x_n), \quad (3.5)$$

By Theorem 3.2 we obtain a known explicit formula for $\det(S)_f$ on chains presented in [7, Corollary 3] and [13, Corollary 1].

Corollary 3.1 *If* $S = \{ x_1, x_2, \ldots\ldots\ldots\ldots x_n \}$ *is a chain, then*

$$\text{Det } (S)_f = f(x_1) \prod_{k=2}^{n} (f(x_k) - f(x_{k-1})) \quad (3.6)$$

Proof : By Theorem 3.2 we have

$\det(S)_f = f_1(x_1)f_2(x_2)\ldots\ldots\ldots f_n(x_n)$, where $f_1 = f$ and

$f_{k+1}(x) = f_k(x) - f_k(x_k) = f(x) - f(x_k)$ for all $k = 1, 2, \ldots\ldots\ldots\ldots, n-1$. This completes the proof.

By Theorem 3.2 we also obtain a known explicit formula for $\det(S)_f$ on *a*-sets. This formula has been presented (with different notation) in [13, Corollary of Theorem 3] and [10,Corollaries 5.1 and 5.2], and also in [16, Theorem 3] in number-theoretic setting.

The case $f(a) = 0$ is trivial, since then $(S)_f = \text{diag}(f(x_1), f(x_2), \ldots\ldots, f(x_n))$ and $\det(S)_f = f(x_1)f(x_2)\ldots\ldots f(x_n)$.

Corollary 3.2 *Let* $S = \{ x_1, x_2, \ldots\ldots\ldots\ldots x_n \}$ *be an a-set, where* $f(a) \neq 0$. *If* $a \in S$ *(i.e.* $a = x_1$*),*
Then $\det(S)_f = f(a)(f(x_2) - f(a))\ldots.(f(x_n) - f(a))$. (3.7)
If $a \notin S$, *then*

$$\det(S)_f = \sum_{k=1}^{n} \frac{f(a)(f(x_1) - f(a))\ldots\ldots(f(x_n) - f(a))}{f(x_k) - f(a)} + (f(x_1) - f(a))\ldots.(f(x_n) - f(a)). \quad (3.8)$$

Example 3.1 Wilson's matrix

Let $(P, \preceq) = (\mathbf{Z}+, |)$ and $S = \{2,3,5\}$.

Then $S = \begin{bmatrix} (2-1)!+1 & (1-1)!+1 & (1-1)!+1 \\ (1-1)!+1 & (3-1)!+1 & (1-1)!+1 \\ (1-1)!+1 & (1-1)!+1 & (5-1)!+1 \end{bmatrix}$. Since S is an A-set with the chain

$A = \{1,1\}$ by (3.1) we have $f_1 = f$, $f_2(x) = f_1(x) - f_1(1)^2/f_1(1)$ and $f_3(x) = f_2(x) - f_2(1)^2/f_2(3)$. and. Let $f(x) = (x-1)!+1$. Then $f_1(x) = (x-1)!+1$, $f_2(x) = (x-1)! + \frac{1}{2}$, $f_3(x) = (x-1)! + \frac{2}{5}$

and by Theorem 3.1 $(S)_f = M^TDM$, where D=diag$(2, \frac{5}{2}, \frac{122}{5})$and M = $\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{5} \\ 0 & 0 & 1 \end{bmatrix}$

and by Theorem 3.2 we have $\det(S)_f = f_1(2)f_2(3)f_3(5) = 2(\frac{5}{2})(\frac{122}{5}) = 122$.

Example 3.1 Reciprocal Wilson's matrix

Let $(P, \preceq) = (\mathbf{Z}+,|)$ and $S = \{2,3,5\}$.

Then S=$\begin{bmatrix} \frac{1}{(2-1)!+1} & \frac{1}{(1-1)!+1} & \frac{1}{(1-1)!+1} \\ \frac{1}{(1-1)!+1} & \frac{1}{(3-1)!+1} & \frac{1}{(1-1)!+1} \\ \frac{1}{(1-1)!+1} & \frac{1}{(1-1)!+1} & \frac{1}{(5-1)!+1} \end{bmatrix}$. Since S is an A-set with the chain $A = \{1,1\}$ by (3.1) we have $f_1 = f$, $f_2(x) = f_1(x) - f_1(1)^2/f_1(2)$ and $f_3(x) = f_2(x) - f_2(1)^2/f_2(3)$.

Let $f(x) = \frac{1}{(x-1)!+1}$. Then $f_1(x) = \frac{1}{(x-1)!+1}$, $f_2(x) = \frac{1}{(x-1)!+1} - 2$, $f_3(x) = \frac{1}{(x-1)!+1} - \frac{7}{5}$

and by Theorem 3.1 $(S)_f = M^TDM$, where D=diag$((\frac{1}{2})\ (\frac{-5}{3})(\frac{-34}{25}))$ and M = $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & \frac{3}{5} \\ 0 & 0 & 1 \end{bmatrix}$

and by Theorem 3.2 we have $\det(S)_f = f_1(2)f_2(3)f_3(5) = \frac{1}{2} \times \frac{-5}{3} \times \frac{-34}{25} = \frac{17}{15}$.

3.3 Inverse of Wilson's and Reciprocal Wilson's meet matrices on *A*-sets

By Structure Theorem we obtain a new recursive formula for $(S_f)^{-1}$ on A-sets.

Theorem 3.3 *Let* $S = \{ x_1, x_2, \ldots\ldots x_n \}$ *be an A-set, where* $A = \{ a_1, a_2, \ldots\ldots a_{n-1} \}$ *is a multichain. Let* $f_1, f_2, \ldots, f_n$ *be the functions defined in* (3.1), *where* $f_i(x_i) \neq 0$ *for* $i = 1,2,\ldots,n$. *Then* $(S)_f$ *is invertible and* $(S_f)^{-1} = N\triangle N^T$ (3.9)

where $\triangle = \text{diag}(1/f_1(x_1), 1/f_2(x_2), \ldots, 1/f_n(x_n))$ *and N is the n×n upper triangular matrix with* 1*'s on its main diagonal, and further*

$$(N)_{ij} = -\frac{f_i(a_i)}{f_i(x_i)} \prod_{k=i+1}^{j-1}\left(1 - \frac{f_k(a_k)}{f_k(x_k)}\right) \qquad (3.10)$$

for all i < j.

Proof: By Structure Theorem $(S)_f = M^TDM$, where M is the matrix defined in (3.3) and $D = \text{diag}\,(f_1(x_1), f_2(x_2), \ldots\ldots, f_n(x_n))$. Therefore $(S_f)^{-1} = N\triangle N^T$, where $D^{-1} = \text{diag}(1/f_1(x_1), 1/f_2(x_2), \ldots, 1/f_n(x_n))$ and $M^{-1} = N$ is the $n \times n$ upper triangular matrix in 3.10.

Example 3.1.1

S is considered the same as in Example 3.1 and by $(S_f)^{-1} = N\triangle N^T$,

then for Wilson's matrix

$\triangle$=diag (1/5,1/12,1/252)) , N= M^{-1}, N= $\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{2}{5} \\ 0 & 1 & -\frac{1}{5} \\ 0 & 0 & 1 \end{bmatrix}$,

$$(S_f)^{-1} = \begin{bmatrix} \frac{37}{61} & -\frac{12}{61} & -\frac{1}{61} \\ -\frac{12}{61} & \frac{49}{122} & -\frac{1}{122} \\ -\frac{1}{61} & -\frac{1}{122} & \frac{5}{122} \end{bmatrix}$$

And for Reciprocal Wilson's matrix,

$$\triangle = \text{diag}\left(2, -\frac{3}{5}, -\frac{25}{34}\right), \quad N = M^{-1}, \quad N = \begin{bmatrix} 1 & -2 & -\frac{4}{5} \\ 0 & 1 & -\frac{3}{5} \\ 0 & 0 & 1 \end{bmatrix},$$

$$(S_f)^{-1} = \begin{bmatrix} -\frac{74}{85} & \frac{72}{85} & \frac{10}{17} \\ \frac{72}{85} & -\frac{147}{170} & \frac{15}{34} \\ \frac{10}{17} & \frac{15}{34} & -\frac{25}{34} \end{bmatrix}$$

Corollary 3.3 *Let* $S = \{x_1, x_2, \ldots\ldots\ldots\ldots x_n\}$ *be an a-set, where* $f(a) \neq 0$ *and* $f(x_k) \neq f(a)$ *for all* $k = 2, \ldots\ldots., n$. *If* $a \in S$ *(i.e.* $a = x_1$*), then* $(S)_f$ *is invertible and*

$$\left((S_f)^{-1}\right)_{ij} = \begin{cases} \frac{1}{f(a)} + \sum_{k=2}^{n} \frac{1}{f(x_k)-f(a)} & if\ i = j = 1, \\ \frac{1}{f(x_k)-f(a)} & if\ 1 < i = j, \\ \frac{1}{f(a)-f(x_k)} & if\ 1 = i < j = k\ or\ 1 = j < i = k \\ 0 & otherwise \end{cases} \qquad (3.11)$$

If $a \notin S$ *and further* $f(x_1) \neq f(a)$ *and* $\frac{1}{f(a)} \neq \sum_{k=1}^{n} \frac{1}{f(x_k)-f(a)}$ *, then* $(S)_f$ *is invertible and*

$$\left((S_f)^{-1}\right)_{ij} = \begin{cases} \frac{1}{f(x_k)-f(a)} - \frac{1}{[f(x_k)-f(a)]^2}\left(\frac{1}{f(a)} + \sum_{k=1}^{n} \frac{1}{f(x_k)-f(a)}\right)^{-1} & if\ i = j, \\ \frac{1}{[f(x_k)-f(a)][f(x_k)-f(a)]}\left(\frac{1}{f(a)} + \sum_{k=1}^{n} \frac{1}{f(x_k)-f(a)}\right)^{-1} & if\ i \neq j. \end{cases} \qquad (3.12)$$

CONCLUSION:

In this paper we prove by examples that the Wilson's Meet and Reciprocal Wilson's Meet matrices on A sets satisfies structure theorem and the determinant and inverse of these matrices can be calculated through results based on A sets.

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