FAIRNESS AND PARADOXES IN VOTING SYSTEMS: A MATHEMATICAL EXPLORATION OF SOCIAL CHOICE MECHANISMS

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Abstract

The design of fair and consistent voting systems remains a central challenge in social choice theory. Despite the intuitive appeal of aggregating individual preferences into a collective decision, various mathematical impossibility results and paradoxes highlight the intrinsic limitations of such processes. This paper presents a formal mathematical analysis of fairness and paradoxes in voting systems, focusing on the implications of Arrow's Impossibility Theorem and the Gibbard–Satterthwaite Theorem, alongside key paradoxes such as Condorcet cycles and the no-show paradox. Employing axiomatic frameworks and formal logic, we explore the conflict between desirable voting criteria—such as monotonicity, strategy-proofness, and independence of irrelevant alternatives—and demonstrate how certain fairness conditions are mathematically incompatible. By modelling voter preferences and outcomes through combinatorial and geometric tools, this study reveals the structural constraints that underlie democratic decision-making, emphasizing the trade-offs inherent in any voting mechanism. The findings underscore the necessity of reconciling idealistic normative goals with the limitations imposed by formal mathematical theorems.

Keywords:

Social choice theory, Arrow's Impossibility Theorem, Gibbard–Satterthwaite Theorem, voting paradoxes, Condorcet paradox, fairness criteria, strategy-proofness, monotonicity, preference aggregation, mathematical voting models

1. Introduction

Voting is a fundamental mechanism through which collective decisions are made in democratic societies. At its core lies a seemingly straightforward goal: to translate individual preferences into a socially acceptable outcome. However, as the pioneering work of Kenneth Arrow (1950, 1951) demonstrated, this goal is fraught with intrinsic mathematical difficulties. Arrow's Impossibility Theorem formalizes the inherent conflict between seemingly reasonable fairness criteria and exposes the impossibility of satisfying all of them simultaneously in a ranked voting system with three or more candidates.

This paper undertakes a mathematically rigorous exploration of fairness and paradoxes in voting systems, grounded in the axiomatic method and combinatorial models. We analyze the implications of key theorems in social choice—particularly Arrow's Theorem and the Gibbard–Satterthwaite Theorem (Gibbard, 1973)—and use these as lenses through which to examine classical voting paradoxes, including the Condorcet cycle (Black, 1958; Gehrlein & Lepelley, 2010) and the monotonicity and participation paradoxes.

Unlike philosophical or political treatments, our approach is strictly mathematical. We define voting rules as functions over the set of all possible voter profiles, and fairness properties as formal constraints on these functions. Through logical deductions and counterexamples, we establish the limitations imposed by these constraints and their interaction with the strategic behavior of voters.

Subsequent sections present formal definitions, theorems, and proofs. Our focus remains on the mathematical foundations: preference profiles, fairness axioms, paradoxical outcomes, and geometric and algebraic tools used to model and analyze them (Saari, 1994; Hodge & Klima, 2005; Taylor & Pacelli, 2008).

2. Preliminaries and Notation

We consider a finite set of voters $V = \{v_1, v_2, \ldots, v_n\}$ and a finite set of alternatives (candidates or options) $A = \{a_1, a_2, \ldots, a_m\}$, with $n \geq 2$ and $m \geq 3$, unless otherwise stated.

Each voter $v_i \in V$ has a **preference ordering** over the set A, which we model as a strict total order: a complete, transitive, and antisymmetric ranking of the elements of A. The set of all such linear orderings of A is denoted by $\mathcal{L}(A)$.

A **preference profile** is a vector $\mathbf{P} = (P_1, P_2, \ldots, P_n) \in \mathcal{L}(A)^n$, where each $P_i \in \mathcal{L}(A)$ represents the preference of voter v_i.

A **social choice function** (SCF) is a function:

$$f : \mathcal{L}(A)^n \to A,$$

which maps each profile $\mathbf{P}$ to a single winning alternative $a \in A$. Alternatively, a **social welfare function** (SWF) returns a complete ranking of all alternatives:

$$F : \mathcal{L}(A)^n \to \mathcal{L}(A).$$

2.1 Axioms of Fairness

To evaluate the fairness and robustness of a voting system, we consider several widely accepted axioms:

- **Unanimity (Pareto Efficiency):** If every voter ranks a above b, then the collective outcome should rank a above b.
- **Non-Dictatorship:** There is no voter v_i whose preferences always determine the social choice regardless of others.
- **Independence of Irrelevant Alternatives (IIA):** The ranking between any two alternatives a and b in the social outcome depends only on the individual rankings of a and b, and not on preferences over other alternatives.
- **Monotonicity:** If a winning candidate a improves their position in one or more voter rankings, then a should not lose.
- **Strategy-Proofness:** No voter can obtain a more preferred outcome by misrepresenting their preferences.

2.2 Voting Rules

Several voting rules (social choice functions) are commonly studied:

- **Plurality Rule:** The alternative with the most first-place votes wins.
- **Borda Count:** Alternatives are scored based on their position in each voter's ranking.
- **Condorcet Method:** An alternative that beats every other alternative in pairwise comparisons (if it exists) is the winner.
- **Runoff Voting, Approval Voting, etc.:** Other systems with distinct mathematical properties.

Each voting rule may satisfy or violate different combinations of the fairness axioms. These interactions form the basis for analyzing paradoxes and impossibility results.

3. Arrow's Impossibility Theorem

One of the foundational results in the theory of social choice is Arrow's Impossibility Theorem (Arrow, 1950; 1951), which rigorously demonstrates the limitations of aggregating individual preferences into a collective decision.

3.1 Statement of the Theorem

Let A be a set of at least three alternatives and V a finite set of voters. Suppose we wish to construct a social welfare function:

$$F : \mathcal{L}(A)^n \to \mathcal{L}(A)$$

that satisfies the following conditions:

1. **Unrestricted Domain (Universal Domain):** The function FFF must produce a complete, transitive ranking for every possible combination of individual preference orders.
2. **Pareto Efficiency (Unanimity):** If every individual prefers a to b, then the social ranking must also prefer a to b.
3. **Independence of Irrelevant Alternatives (IIA):** The social preference between a and b should depend only on the individual preferences between a and b, regardless of other alternatives.
4. **Non-Dictatorship:** There exists no voter $v_i \in V$ such that F always reflects the strict preference order of v_i regardless of the preferences of others.

Arrow's Theorem states:

There exists no social welfare function F satisfying all four conditions simultaneously when $|A| \geq 3$.

3.2 Proof Sketch

The proof, while elegant and subtle, follows by contradiction. Assume such a function F exists. By considering specific profiles and applying IIA and unanimity conditions recursively, one constructs a sequence of logical deductions that force the social ranking to obey the preferences of a single voter across all inputs—thereby violating the non-dictatorship condition.

The proof involves:

- Constructing profiles where preferences between two alternatives are constant while others change.
- Showing that if IIA and unanimity are upheld, then the preferences of one voter must dictate the outcome on all pairwise comparisons.
- Concluding that this voter must be a dictator, contradicting the assumption.

For formal derivations, see Arrow (1951), Saari (2001), and Taylor & Pacelli (2008).

3.3 Implications

Arrow's theorem has far-reaching implications:

- **Normative Limitations:** It demonstrates that no voting system can be perfectly fair by satisfying all seemingly reasonable conditions.
- **Design Trade-offs:** Voting systems must sacrifice at least one of the axioms, leading to different approaches in political, economic, and computational systems.

- **Mathematical Rigor in Policy:** The theorem illustrates the power of mathematical abstraction in revealing deep truths about collective decision-making.

Arrow's result laid the foundation for later impossibility theorems, such as the Gibbard–Satterthwaite Theorem, which we explore next.

4. The Gibbard–Satterthwaite Theorem

The Gibbard–Satterthwaite Theorem (Gibbard, 1973; Satterthwaite, 1975) is another cornerstone result in the mathematical theory of voting. While Arrow's theorem concerns social welfare functions that return complete rankings, the Gibbard–Satterthwaite theorem applies to social choice functions, which return a single winning alternative.

4.1 Statement of the Theorem

Let A be a set of at least three alternatives, and V a set of at least two voters. Consider a social choice function:

$$F : \mathcal{L}(A)^n \to \mathcal{L}(A)$$

that satisfies the following conditions:

1. **Onto (Surjectivity):** Every alternative a ∈A is the output of f for some preference profile $\mathbf{P} \in \mathcal{L}(A)^n$.
2. **Strategy-Proofness:** No voter can achieve a more preferred outcome by misrepresenting their preferences.
3. **Non-Dictatorship:** There is no voter whose top-ranked alternative is always selected by f regardless of other preferences.

Gibbard–Satterthwaite Theorem:

Every non-dictatorial, strategy-proof social choice function with at least three alternatives is necessarily manipulable—that is, strategy-proofness implies dictatorship.

4.2 Interpretation and Consequences

This theorem complements Arrow's result by addressing manipulability rather than consistency of aggregation:

- If a voting system always selects an outcome that truly reflects sincere preferences, then it must give one voter decisive power—violating fairness.
- Conversely, if no one voter has total control, then at least one voter can sometimes benefit from voting strategically.

This result formalizes the intuition that strategic voting is unavoidable in general settings and motivates the study of mechanism design and restricted domains (such as single-peaked preferences), where manipulation can be avoided.

4.3 Sketch of the Proof

The proof involves:

- Fixing the preferences of all voters except one.
- Considering how the output of the social choice function changes as the remaining voter varies their preferences.

- Showing that the only way to avoid strategic advantage in such a setting is if that voter's top choice always wins—establishing dictatorship.

The result is general and powerful, applying to any deterministic rule with full domain and three or more options.

4.4 Comparison with Arrow's Theorem

Feature	Arrow's Theorem	Gibbard–Satterthwaite Theorem
Type of Function	Social Welfare Function	Social Choice Function
Output	Complete ranking	Single winning alternative
Core Assumption Violated	IIA or Non-Dictatorship	Strategy-Proofness or Non-Dictatorship
Implication	**No fair aggregation rule exists**	**All strategy-proof rules are dictatorial**

These two theorems form the backbone of modern mathematical social choice theory and illustrate deep, inherent conflicts in designing fair and strategy-proof voting systems.

5. Voting Paradoxes and Condorcet Efficiency

While Arrow's and Gibbard–Satterthwaite's theorems highlight axiomatic limitations in social choice mechanisms, the study of voting paradoxes explores the structural irregularities that emerge in specific voting scenarios. One central theme is the Condorcet Paradox, where majority preferences can be cyclic and intransitive, even if individual preferences are rational.

5.1 The Condorcet Paradox

Let A={a,b,c} and suppose a group of voters express the following strict preference rankings:

Voter Group	Preference Order
1/3	$a \succ b \succ c$
1/3	$b \succ c \succ a$
1/3	$c \succ a \succ b$

In pairwise comparisons:

- a beats b: $a \succ b$ (2/3 of voters)
- b beats c: $b \succ c$ (2/3 of voters)
- c beats a: $c \succ a$ (2/3 of voters)

This creates a cycle: a≻ b≻ c, violating transitivity. No clear winner exists, contradicting the idea that majority preferences always produce coherent group outcomes. This paradox undermines the Condorcet Principle, which states:

If an alternative defeats every other alternative in pairwise comparisons, it should be the winner.

5.2 Condorcet Efficiency

Despite the paradox, many voting rules try to elect the Condorcet winner when one exists. Condorcet efficiency is defined as:

$$\text{Condorcet Efficiency} = \frac{\text{Number of profiles where a rule elects the Condorcet winner}}{\text{Number of profiles with a Condorcet winner}}$$

Let R be a voting rule and P the set of all profiles with a Condorcet winner. Then:

$$\mathrm{CE}(R) = \frac{|\{p \in P : R(p) = CW(p)\}|}{|P|}$$

Where CW(p) is the Condorcet winner at profile p. High Condorcet efficiency is desirable but often comes at the cost of violating other properties (like monotonicity or strategy-proofness).

5.3 Empirical and Theoretical Evaluations

Gehrlein and Lepelley (2010), and later Diss and Gehrlein (2015), provided extensive statistical analysis of the Condorcet efficiency for common voting rules:

Voting Rule	Approx. Condorcet Efficiency (3 candidates, many voters)
Plurality	~64%
Borda Count	~88%
Runoff (Two-Round)	~75%
Copeland	100% (when CW exists)
Minimax	100%

The efficiency depends on the number of candidates and voters. Simulation methods are often used to estimate these values under Impartial Culture (IC) or Impartial Anonymous Culture (IAC) models (Merrill & Grofman, 1999).

5.4 Other Paradoxes

- **No-show paradox**: A voter may hurt their preferred outcome by voting.
- **Monotonicity paradox**: Raising a candidate in preference order can cause them to lose.
- **Reversal paradox**: Reversing all preferences can still elect the same winner.

These paradoxes reflect the nonlinear and often chaotic nature of collective decision-making. Saari (1994, 2001) used geometrical tools to visualize and explain these inconsistencies, offering a new lens on the internal structure of voting systems.

5.5 Mathematical Representation of Cycles

We can represent the majority preference relation as a directed graph G = (A, E), where an edge a→b if a beats b in a majority vote. The Condorcet paradox implies that G may contain directed cycles:

- If G is acyclic, the Condorcet winner corresponds to a source node.
- If G is cyclic, no Condorcet winner exists.

Detecting such cycles becomes a computational task, linking voting theory with graph theory and algorithm design.

6. Combinatorial Models and Voting Rules

The mathematical structure of voting systems lends itself naturally to combinatorics, where one studies the arrangement, counting, and optimization of preference profiles, voting outcomes, and paradoxical scenarios. This section formalizes voting rules and their behavior using combinatorial tools.

6.1 Preference Profiles and Permutations

Let A= {a_1, a_2..., a_m} be a finite set of m candidates. Each voter's strict preference can be represented as a linear order, i.e., a permutation $\pi \in S_m$, where S_m is the symmetric group of all m! permutations. A preference profile for n voters is a tuple P= (π_1, π_2 …, π_n) ∈(S_m).

The Impartial Culture (IC) model assumes each voter selects a ranking independently and uniformly at random from S_m . This forms the basis for probabilistic evaluations of voting rules, especially in computing average-case behavior and paradox likelihoods.

6.2 Voting Rules as Aggregation Functions

A voting rule is a function:

$$f : (S_m)^n \to A,$$

which maps each profile to a single winner. Alternatively, **social welfare functions** return a full ranking:

$$F : (S_m)^n \to S_m.$$

Examples of common positional scoring rules include:

- **Plurality**: Score $s_i = 1$ for top-ranked candidate, 0 otherwise.
- **Borda**: Score $s_i = m - r_i$ for a candidate ranked r_i-th by a voter.
- **Veto**: Each candidate gets 1 point unless ranked last.

The score vector s= (s_1, s_2,…,s_m) for each candidate is computed across voters, and the candidate with the highest total wins.

6.3 Counting Paradoxes and Profiles

Let $N(m, n) = (m!)^n$ be the number of possible profiles with m candidates and n voters.

For example, with $m = 3$ and $n = 3$, we have $6^3 = 216$ total profiles. Combinatorially:

- The number of Condorcet paradox profiles (with cyclic majority) can be counted using tournament theory.
- The number of non-manipulable profiles under certain rules can be bounded using strategy-proofness criteria.

This gives rise to quantitative analysis of how frequently paradoxes or inefficiencies occur.

6.4 Algebraic and Geometric Representations

Saari (2001, 2008) proposed geometric methods for visualizing the behavior of voting rules:

- Each voter ranking corresponds to a basis vector in Rm!. A profile is a point in the preference simplex, where coordinates represent frequencies of rankings.
- Voting rules define hyperplanes or regions in this space where one candidate wins.

These geometric representations clarify the instability of certain rules and allow one to visualize how small changes in preferences can flip outcomes—providing insight into paradoxes and anomalies.

6.5 Connections with Group Theory and Symmetry

Voting rules often respect certain **symmetries**:

- **Anonymity**: All voters are treated equally.
- **Neutrality**: All candidates are treated equally.
- **Reversal symmetry**: Reversing preferences inverts the outcome ranking.

These properties correspond to invariant subgroups in $S_n \times S_m$ and their preservation or violation reflects important fairness and consistency traits.

Cameron (2001) and Taylor & Pacelli (2008) show how group actions and orbits classify rules and preference transformations, grounding social choice in pure algebraic combinatorics.

7. Domain Restrictions and Constructive Outcomes

While fundamental theorems like those of Arrow and Gibbard–Satterthwaite underscore the limitations of social choice mechanisms in unrestricted domains, introducing structured constraints on voter preferences can yield more favorable results. These domain restrictions narrow the range of admissible preference profiles, enabling voting systems to operate without encountering many of the paradoxes and strategic issues common in unrestricted settings. This section examines key domain restrictions and their positive implications for the design of fair and efficient voting systems.

7.1 Single-Peaked Preferences

A pivotal domain restriction is single-peakedness, where voter preferences can be organized along a linear spectrum, and each voter has a most-preferred option (a peak), with preference decreasing symmetrically as alternatives move further from this peak.

Mathematically, for a candidate set $A = \{a_1, a_2, ..., a_m\}$ ordered linearly as $a_1 \prec a_2 \prec \cdots \prec a_m$, a preference profile is single-peaked if every voter's ranking has a unique top choice, and their preferences drop off on either side.

Implication: In this structured domain, voting rules like Borda Count, Plurality, and Condorcet-based systems tend to avoid anomalies such as the Condorcet paradox. The Median Voter Theorem further asserts that the candidate closest to the median voter's peak will win under majority rule, yielding stable and predictable outcomes.

7.2 Single-Crossing Preferences

Another influential restriction is the single-crossing property, where voters can be ordered such that, for any pair of alternatives, the preference switch between them happens at most once across the ordering.

Formally, for candidates a and b, if voters are arranged in a sequence, the order of preference between a and b changes no more than once from one voter to the next.

Implication: Single-crossing domains ensure that majority-based methods and Borda Count function efficiently and predictably. The threshold rule is also effective here, and many paradoxes present in general domains are avoided.

7.3 Unimodal (Top-Order) Preferences

A generalization of single-peakedness is the concept of unimodal preferences, where each voter has a single most-preferred alternative, and their preferences decline as options deviate from this ideal. This allows for more flexible, possibly multi-dimensional interpretations. For instance, a voter ranking $a_1 \succ a_2 \succ a_3$ suggests that a_2 lies at the peak.

Implication: In such settings, common voting methods (Plurality, Borda, Condorcet-based rules) tend to yield more robust outcomes, and systems like Single Transferable Vote (STV) perform well under unimodal distributions by promoting proportional representation and reducing strategic behavior.

7.4 Approval Voting in Structured Domains

Approval voting enables voters to endorse any number of candidates they find acceptable, with the highest approval total determining the winner. It accommodates broader and less rigid preference structures.

Implication: In settings lacking strict single-peaked or single-crossing assumptions, approval voting can still yield compromise winners and limit the spoiler effect. Particularly in multi-winner elections, it offers a nuanced view of voter sentiment, often outperforming Plurality in fairness and strategic resistance.

7.5 Strategy-Proofness in Restricted Domains

A key insight is that manipulability—the incentive to vote insincerely—is closely linked to the structure of the preference domain. When voters' preferences follow constrained forms like single-peakedness, many voting rules become strategy-proof.

For example, while the Borda count is vulnerable to manipulation in general, it resists strategic distortion in a single-peaked domain. Similarly, Condorcet-based systems are more likely to identify sincere majority preferences when the domain is structured.

7.6 Mechanism Design and the Power of Restrictions

Mechanism design aims to create voting procedures that inherently discourage manipulation. When preferences are restricted to certain domains, it becomes possible to construct voting rules that uphold key principles—such as fairness, efficiency, and non-manipulability—simultaneously.

These constraints can enhance the performance of Condorcet-based methods and offer practical guidelines for electoral system design. For instance, policymakers may structure elections so that only preference profiles aligning with single-peaked behavior are admitted, reducing instability and paradoxes in results.

8. Conclusion

This study has examined the delicate relationship between fairness and paradoxes within voting systems from a strictly mathematical standpoint. By analyzing core results such as Arrow's Impossibility Theorem and the Gibbard–Satterthwaite Theorem, we have highlighted the intrinsic constraints that emerge in the absence of restrictions on voter preferences. These foundational results reveal the conflict between fairness, efficiency, and resistance to manipulation in collective decision-making.

Nonetheless, we have shown that imposing certain domain constraints—like single-peaked, single-crossing, or unimodal preferences—can mitigate these challenges. Within such structured domains, it becomes feasible to design voting rules that are both less susceptible to paradoxes and more resistant to strategic manipulation, thereby promoting greater stability in social choice systems.

Additionally, the use of combinatorial approaches and group-theoretic tools has demonstrated how mathematical structures can reveal underlying patterns and properties in voting behavior. These frameworks deepen our understanding of the geometric and algebraic nature of voting systems and provide insight into why paradoxes so frequently arise in practice.

Future Work

Although domain restrictions present viable solutions to the problems discussed, they inevitably reduce the universality of the voting models. Future research may aim to identify methods that preserve robustness without unduly constraining the expressiveness of voter preferences. Exploring advanced concepts in mechanism design may also uncover novel voting rules that remain fair and manipulation-resistant in broader settings.

In conclusion, a rigorous mathematical examination of social choice mechanisms not only clarifies the theoretical limitations of current systems but also highlights the necessary trade-offs between formal rigor and practical relevance in designing fair and functional democratic processes.

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