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REMARKS ON *t*- EVOLUTION OF DGLAP EQUATIONS AT SMALL *x*

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Abstract: We show that the coupled DGLAP equations at small *x* obtained from Taylor approximation method in general yields two inequivalent *t*-evolution of singlet and gluon even though they start with the same identical non- perturbative input. The additional assumption is that in certain *x* and *Q*2 range an analytical relationship between singlet and gluon is possible.

Keywords: *DGLAP equations, singlet distribution, gluon distribution*

1. Introduction

The DGLAP equations [1-5] are the essential tools to calculate the evolution of the parton distributions and the structure functions of the nucleon and nuclei. These equations are integro-differential equations whose complexity rises as we move to higher and higher order. These equations are normally solved numerically independent of data [6] then applied to data to perform the fits. But alternatively, the evolution equations can also be solved analytically under certain assumptions valid to be at low x [8-12] . In it one makes a Taylor approximation and convert the integro-differential DGLAP equations into first order partial differential equations in both *x* and *t*, where *x* is Bjorken variable and $t = \ln\left(\frac{Q^2}{\Lambda^2}\right)$ and Λ as QCD scale parameter. As a second step, an ansatz relating gluon and singlet structure function is assumed [8,9,12] to make them analytically solvable. The present short paper reports an interesting observation: the Taylor approximated DGLAP equations can in general give two inequivalent forms of t-evolution for $F_2^S(x,t)$ and gluon $G(x,t)$ even if they evolve from the same non-perturabtive input at $t = t_0$. In section 2, we outline the formalism and section 3 is devoted to analytical results and section 4 is conclusions.

2. Formalism

Coupled DGLAP equation at leading order (LO) and Taylor approximation method.

The standard form of the coupled DGLAP equations for singlet structure function and gluon distribution function at LO are given as

$$\frac{\partial}{\partial t}F_2^S(x,t)=\frac{a_s(t)}{2\pi}[\{3+4\ln(1-x)\}F_2^S(x,t)+2\int_x^1\frac{dz}{1-z}\left\{(1+z^2)F_2^S\left(\frac{x}{z},t\right)-2F_2^S(x,t)\right\}\text{¿}\frac{+3}{2}N_f\int_x^1 dz\{z^2+(1-z)^2\}G\left(\frac{x}{z},t\right)]$$

$$\frac{\partial}{\partial t}G(x,t)=\frac{3a_s(t)}{\pi}[\left\{\frac{11}{12}-\frac{N_f}{18}+\ln(1-x)\right\}G(x,t)+\int_x^1 dz\frac{zG\left(\frac{x}{z},t\right)-G(x,t)}{1-z}\text{¿}+\int_x^1 dz\left\{z(1-z)+\frac{1-z}{z}\right\}G\left(\frac{x}{z},t\right)+\frac{2}{9}\int_x^1 dz\frac{1+(1-z)^2}{z}F_2^S\left(\frac{x}{z},t\right)]$$

where, N_f is the number of quark flavours , α_s is the strong coupling constant given in LO by $\alpha_s(Q^2)=\frac{12\pi}{(33-2N_f)\ln(Q^2/\Lambda^2)}$. The other constants are $T_R=\frac{1}{2}, C_A=3, C_F=\frac{4}{3}$.

The singlet quark distribution is defined as $\Sigma(x,t)\text{□}\sum_i^{2N_f}\{q_i(x,t)+\acute{q}_i(x,t)\}$, where q_i and $\acute{q}_i$ are quarks and antiquarks of flavour i. So singlet structure function is $F_2^S(x,t)=x\Sigma(x,t)$ and gluon distribution is $G(x,t)=xg(x,t)$.

Taylor approximated form of $F_2^S\left(\frac{x}{z},t\right)$ and $G\left(\frac{x}{z},t\right)$ on the right hand sides of eqn(1) and (2) for small x and taking only upto O(x) will give

$$F_2^S\left(\frac{x}{z},t\right)=F_2^S(x,t)+x\frac{(1-z)}{z}\frac{\partial}{\partial x}F_2^S(x,t) \quad (3)$$

$$G\left(\frac{x}{z},t\right)=G(x,t)+x\frac{(1-z)}{z}\frac{\partial}{\partial x}G(x,t) \quad (4)$$

Equation (1) becomes as in [12] is:

$$t\frac{\partial}{\partial t}F_2^S(x,t)=A_f[\{4\ln(1-x)+2x\}F_2^S(x,t)+\{2x\ln(1/x)+x\}\frac{\partial}{\partial x}F_2^S(x,t)+N_f\left\{1-\frac{3x}{2}\right)G(x,t)\text{¿}\frac{+N_f}{2}\{3x\ln(1/x)-5x\}\frac{\partial}{\partial x}G(x,t)]$$
(5)

Similarly we obtain the Taylor approximated form of gluon DGLAP equation as:

$$t\frac{\partial}{\partial t}G(x,t)=9A_f[\left\{\frac{11}{12}-\frac{N_f}{18}+\ln(1-x)-1+x-\frac{5}{6}+\ln(1/x)+x\right)G(x,t)+\text{¿}\left\{x+1-2x\ln(1/x)+\frac{x}{3}\right)\frac{\partial}{\partial x}G(x,t)+\frac{2}{9}\left\{2\ln(1/x)+\frac{1}{2}-2+2x\right)F_2^S(x,t)\text{¿}\frac{+2}{9}\left\{4x\ln(1/x)+\frac{5x}{2}+2-2x\right)\frac{\partial}{\partial x}F_2^S(x,t)]$$
(6)

Where, $A_f=\frac{4}{3\beta_0}$ and $\beta_0=\frac{33-2N_f}{3}$ is LO QCD beta function.

To solve the above equations (5) and (6) one need to assume an explicit relation between singlet and gluon distribution. Generally, exact analytical solution of coupled DGLAP equations or an explicit relation between quark and gluon distributions are not possible. Numerical methods are necessary. However, if one assumes that at small x in a certain Q^2 range, such analytical form of gluon and quark distribution are possible, the most general form can be written as,

$$G(x,t)=xg(x,t)=x\sum_{i=1}^{2N_f}K_iC_i(x,t)\{q_i(x,t)+\acute{q}_i(x,t)\} \quad (7)$$

Which takes into account the flavor independency of gluon. If the coefficients K_i and $C_i(x,t)$ are flavor independent then the expression can be written as

$$G(x,t)=xK\,C(x,t)\sum_{i=1}^{2N_f}\{q_i(x,t)+\acute{q}_i(x,t)\}=K\,C(x,t)F_2^S(x,t) \quad (8)$$

Here, $K\,C(x,t)$ represent the ratio of the quark and gluon distribution and in general not factorizable in x and t. Putting in equation (5) and (6) we get,

$$t\frac{\partial}{\partial t}F_2^S(x,t)=A_f[\left\{4\ln(1-x)+2x+N_f\left(1-\frac{3x}{2}\right)K\,C(x,t)+\frac{N_f}{2}(3x\ln(1/x)-5x)\frac{\partial}{\partial x}K\,C(x,t)\right).\text{¿}F_2^S(x,t)+\left\{2x\ln(1/x)+x+\frac{N_f}{2}\{3x\ln(1/x)-5x\}K\,C(x,t)\right)\frac{\partial}{\partial x}F_2^S(x,t)]$$
(9)

$$tK\,C(x,t)\frac{\partial}{\partial t}F_2^S(x,t)=[9A_f(\left(\frac{11}{12}-\frac{N_f}{18}+\ln(1-x)-1+x-\frac{5}{6}+\ln(1/x)+x\right)K\,C(x,t)\text{¿}+\left\{1-2x\ln(1/x)+\frac{4x}{3}\right)\frac{\partial}{\partial x}K\,C(x,t)+\frac{2}{9}\left\{2\ln(1/x)+\frac{1}{2}-2+2x\right\})-t\frac{\partial}{\partial t}K\,C(x,t)]F_2^S(x,t)\text{¿}$$
(10)

3. Analytical treatment

In the earlier work the t-evolution of equation (9) was assumed to be identical to that of (10) and was never studied independently [8]. In this work we now demonstrate that t-evolution obtained from those equations are in general not identical given the same non-perturbative input at $t=t_0$. This feature overlooked by us earlier might have theoretical and phenomenological consequences in general.
To proceed further, we solve the above two equations (9) and (10) by Lagrange's Auxiliary method [13] and put them in the following forms

$$Q_1(x,t)\frac{\partial}{\partial t}F_2^S(x,t)+P_1(x,t)\frac{\partial}{\partial x}F_2^S(x,t)=R_1(x,t)F_2^S(x,t)$$

$$Q_2(x,t)\frac{\partial}{\partial t}F_2^S(x,t)+P_2(x,t)\frac{\partial}{\partial x}F_2^S(x,t)=R_2(x,t)F_2^S(x,t)$$

Where, $Q_1(x,t), P_1(x,t), R_1(x,t), Q_2(x,t), P_2(x,t), R_2(x,t)$ are explicit functions of x,t as given in [12]. The solutions of equations (11) and (12) are obtained by solving the following auxiliary system of ordinary differential equations,

$$\frac{dx}{P_m(x,t)}=\frac{dt}{Q_m(x,t)}=\frac{dF_2^S(x,t)}{R_m(x,t)F_2^S(x,t)} \qquad (13)$$

Where, m runs from 1 to 2 referring respectively the two equations (11) and (12). If $u_m(x,t)=C_m$ and $v_m(x,t,F_2^S(x,t))=D_m$ are two independent solutions of the above equations (11) and (12), then in general their solutions are [13]

$$f_m(u_m, v_m)=0 \qquad (14)$$

where, f_m are arbitrary functions of u_m and v_m.
Solving the auxiliary equations, we get

$$u_1(x,t)=t\,X_1(x,t)$$
$$v_1(x,t,F_2^S(x,t))=F_2^S(x,t)Y_1(x,t) \qquad (15),(16) \text{ respectively}$$

And,

$$u_2(x,t)=t\,X_2(x,t)$$
$$v_2(x,t,F_2^S(x,t))=F_2^S(x,t)Y_2(x,t) \qquad (17),(18) \text{ respectively}$$

Where, $X_1(x,t), X_2(x,t), Y_1(x,t), Y_2(x,t)$ are as follows:

$X_1(x,t) = \exp[(1/A_f)\int dx / \{2x\ln(1/x) + x + (N_f/2)\{3x\ln(1/x) - 5x\}KC(x,t)\}]$

$Y_1(x,t) = \exp[\int\{4\ln(1-x) + 2x + N_f(1 - 3x/2)KC(x,t) + (N_f/2)(3x\ln(1/x) - 5x)\,\partial KC(x,t)/\partial x\} / \{2x\ln(1/x) + x + (N_f/2)\{3x\ln(1/x) - 5x\}KC(x,t)\}dx]$

$X_2(x,t) = \exp[\int KC(x,t) / \{9A_f[(1 - 2x\ln(1/x) + 4x/3)KC(x,t) + (2/9)(2 - 2x - 4x\ln(1/x) + 5x/2)]\}dx]$

$$Y_2(x,t)=\exp\left(\int\frac{9A_f\left\{\left(\frac{11}{12}-\frac{N_f}{18}+\ln(1-x)+2x-\frac{11}{6}+\ln\left(\frac{1}{x}\right)\right)KC(x,t)+\left(1-x\ln\left(\frac{1}{x}\right)+\frac{x}{3}\right)\frac{\partial}{\partial x}KC(x,t)+\frac{2}{9}\left(2\ln\left(\frac{1}{x}\right)-\frac{3}{2}+2x\right)\right\}-t\frac{\partial}{\partial t}KC(x,t)}{9A_f\left(1-2x\ln(1/x)+\frac{4x}{3}\right)KC(x,t)+\frac{2}{9}\left(2-2x-4x\ln(1/x)+\frac{5x}{2}\right)}dx\right)$$

(19)

The simplest possibility of finding a unique expression for $F_2^S(x,t)$ to satisfy the general solution (14) is the linear combination of u_m and v_m in $F_2^S(x,t)$ such as,

$$u_m+\alpha v_m=\beta \qquad (20)$$

Where, α and β are two unknown quantities. Considering $m=1$ and putting the expressions for $u_1(x,t)$ and $v_1(x,t,F_2^S(x,t))$ as in equations (15) and (17) in the above equation(20) we get the corresponding solution as

$$F_2^{S(I)}(x,t)=\frac{1}{\alpha Y_1(x,t)}\left[\beta-tX_1(x,t)\right) \quad (21)$$

Similarly for m=2 we obtain

$$F_2^{S(II)}(x,t)=\frac{1}{\alpha Y_2(x,t)}\left[\beta-tX_2(x,t)\right) \quad (22)$$

Therefore the ratio becomes,

$$R(x,t)=\frac{F_2^{S(I)}(x,t)}{F_2^{S(II)}(x,t)}=\frac{Y_2(x,t)}{Y_1(x,t)}\left[\frac{\beta-tX_1(x,t)}{\beta-tX_2(x,t)}\right) \quad (23)$$

At certain $t=t_0$, we use the boundary condition $F_2^{S(I)}(x,t_0)=F_2^{S(II)}(x,t_0)=F_2^S(x,t_0)$ and hence, we can express β from equation (21) and (22) in terms of $X_1(x,t), Y_1(x,t), X_2(x,t)$ and $Y_2(x,t)$ as

$$\beta=t_0\left[\frac{X_1(x,t_0)Y_2(x,t_0)-X_2(x,t_0)Y_1(x,t_0)}{Y_2(x,t_0)-Y_1(x,t_0)}\right) \quad (24)$$

Correspondingly, we obtain the following expressions for $F_2^{S(I)}(x,t)$ and $F_2^{S(II)}(x,t)$ as

$$F_2^{S(I)}(x,t)=F_2^S(x,t_0)\left(\frac{t}{t_0}\right)\frac{Y_1(x,t_0)}{Y_1(x,t)}\left[\frac{X_1(x,t)-\frac{\beta}{t}}{X_1(x,t_0)-\frac{\beta}{t_0}}\right) \quad (25)$$

$$F_2^{S(II)}(x,t)=F_2^S(x,t_0)\left(\frac{t}{t_0}\right)\frac{Y_2(x,t_0)}{Y_2(x,t)}\left[\frac{X_2(x,t)-\frac{\beta}{t}}{X_2(x,t_0)-\frac{\beta}{t_0}}\right) \quad (26)$$

With ratio as

$$R(x,t)=\frac{\frac{Y_1(x,t_0)}{Y_1(x,t)}\left[\frac{X_1(x,t)-\frac{\beta}{t}}{X_1(x,t_0)-\frac{\beta}{t_0}}\right)}{\frac{Y_2(x,t_0)}{Y_2(x,t)}\left[\frac{X_2(x,t)-\frac{\beta}{t}}{X_2(x,t_0)-\frac{\beta}{t_0}}\right)} \quad (27)$$

This is the main result of the work : where as in standard DGLAP eqautions, a single non- perturbative input is sufficient to determine the t-evolution of the structure function. The solutions of the Taylor approximated version yields two alternative forms of t-evolution if the singlet and gluon distribution are related analytically by equation(8). Only under special conditions of

a) $C(x,t)$ is either constant or t-independent function and

b) the factor β vanishes, the two evolutions will be identical.

To proceed further, we need analytical form of $C(x,t)$. There are several theoretical and phenomenological studies on the plausible structure of $C(x,t)$. In 1980s, Lopez and Yndurain suggested the behaviour of $F_2^S(x,t)$ and $G(x,t)$ at LO and at small x region. At $x \to 0$, it was observed that

$$F_2^S(x,Q^2)_{x\to 0} \to B_S(Q^2)x^{-\lambda_s} \tag{28}$$

$$G(x,Q^2)_{x\to 0} \to B_G(Q^2)x^{-\lambda_G} \tag{29}$$

As $\lambda_S = \lambda_G$, B_S and B_G are Q^2 dependent function and λ_S is strictly positive. So from equation (28) and (29) one can derive the relation between gluon and singlet distribution as Q^2 dependent which has been more recently used in [15] . That is

$$\frac{G(x,t)}{F_2^S(x,Q^2)_{x\to 0}} \to \frac{B_G(Q^2)}{B_S(Q^2)} \to f(Q^2) \tag{30}$$

It is also well known that as $x \to 0, Q^2 \to \propto$ DGLAP equations can determine the Parton Distribution Functions solely, provided at the input scale, their behaviour is sufficiently soft. In this limit, the Parton Distribution Functions exhibit a Double Asymptotic Scaling (DAS) [16-20]. Specifically as $x \to 0, Q^2 \to \propto$,the singlet and gluon sector grows as

$$x\Sigma(x,Q^2) \to N_\Sigma \frac{\gamma}{\rho}\frac{1}{\sqrt{4\pi\gamma\sigma}}e^{2\gamma\sigma-\delta\sigma/\rho}$$

$$x\,g(x,Q^2) \to N_g \frac{1}{\sqrt{4\pi\gamma\sigma}}e^{2\gamma\sigma-\delta\sigma/\rho}$$

(31), (32) respectively

Where γ and δ are defined as

$$\gamma = \left(\frac{12}{\beta_0}\right)^{1/2}, \delta = \left(11 + \frac{2N_f}{27}\right)/\beta_0 \tag{33}$$

And double scaling variables σ and ρ are defined as

$$\sigma \equiv \left[\ln\left(\frac{x_0}{x}\right)\ln\frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)}\right)^{1/2}$$

$$\rho \equiv \left[\frac{\ln(x_0/x)}{\ln\{\ln(Q^2/\Lambda^2)/\ln(Q_0^2/\Lambda^2)\}}\right)^{1/2}$$

(34) ,(35) respectively

Which leads to,

$$C(x,t) \approx \left[\frac{\ln(1/x)}{\ln t}\right)^{\frac{1}{2}} \tag{36}$$

Equation (30) and (36) will thus yield two inequivalent t-evolution even if we set $\beta = 0$. At the phenomenological level, a crude assumption like $C(x,t)$ is both x and t independent also pursued [9] and found the phenomenological range of constant K from data.

On the other hand in some references [21,22], $C(x,Q^2)$ is only assumed to be x dependent and the effective form of $C(x) \to ax^b, ce^{dk}$ (where a,b,c,d,k are constants) are taken. In such cases if $\beta = 0$, then the two alternative t-evolutions will be identical. But the theoretical limitation of such t-independent $C(x,Q^2)$ has been pointed recently in [23] .
For completeness we also mention the quasiclassical solution of the basic GLR equation [24], Kim and Ryskin [25] , obtaining the following result as,

$$F_2^S(x,t) \approx C\frac{\gamma}{4\pi}\sum_f e_f^2 G(x,t) \tag{37}$$

With $C \approx 1.4$ [26] and e_f^2 denote the charge squared of the $f-$ flavoured quark, leading to $C(x, Q^2)$ to be x and Q^2 independent which resembles $C(x, t)$ to be constant (*K*) [9].

4. Conclusion

The DGLAP equations are in general solved numerically valid in all x and Q^2. As one cannot obtain the analytical solution at the fundamental level the relationship between the singlet $F_2^S(x, t)$ and the gluon distribution $G(x, t)$ does not exist in general. But if one assumes a plausible analytical relationship between them, one can have an analytical solution for both the singlet structure function and gluon distribution. However the choice of the relationship is not unique one can in general obtain different analytical forms of the singlet and gluon distributions as well as each having different range of validity in x and Q^2. Further to solve these equations one also has to make choice of the method. Using Lagrange's Auxiliary method, the Taylor approximated DGLAP equations can in general give two inequivalent forms of t-evolution for $F_2^S(x, t)$ and gluon $G(x, t)$ even if they evolve from the same non-perturabtive input at $t = t_0$.

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