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Cubic Spline Approximation Algorithm

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Abstract: In this research paper we had obtained a cubic spline function to interpolate a m^{th} degree piecewise polynomial function, with continuity of class k. We applied Bickle's method to obtain above m^{th} degree spline. We had shown that this spline function exists uniquely and is convergent too. We had also proved that this spline function can be used for solving a second order differential equation for two point boundary value problem. Our reasearch paper consists of 3 sections; in first section we construct the cubic spline which can approximate the given polynomial, in second section we had shown the convergency of the spline function and in the third section we had shown the application of this spline function for solving a two point boundary value problem of second order differential equation.

IndexTerms - Spline Function, Bickley's Method,Boundary value problem, Lacunary Interpolation.

INTRODUCTION

There are several methods which are used to solve a second order differential equation for their approximate solution for equally spaced value of independent variables. For two point boundary value problem, W.Bickley derived a better approximate solution experimentally with the help of a cubic spine function. The objective of research paper is the studies of theoretical and practical aspects of Bickley's methed, and to use these aspects to obtain the approximate solution of two point boundary value problems.

To advance we require the given consistency relationship for all spline functions in $\left(S_m\ ,\ \mathrm{c}^{\mathrm{m-1}}\right)$ with uniform knots i.e. $x=\nu h,\ \nu$=1, 2, 3, 4....., (n-1).

Theorem 1: For any spline function $s(x)\in\left(S_m,\ \mathrm{C}^{\mathrm{m-1}}\right),\ \mathrm{m}\geq 3,$ at the node points the function value and the first derivatives and the function value and the second derivative are given by,

$$\sum_{\nu=0}^{m-1} a_{\nu}^{(m)}\ s(\nu h)\ =\ h\sum_{\nu=0}^{m-1} b_{\nu}^{(m)}\ s'(\nu h), \tag{1.1}$$

$$\sum_{\nu=0}^{m-1} \mathrm{c}_{\nu}^{(m)}\ s(\nu h)\ =\ h^2\sum_{\nu=0}^{m-1} b_{\nu}^{(m)}\ s''(\nu h), \tag{1.2}$$

Whose coefficients are:-

$$a_{\nu}^{(m)} = (\mathrm{m}\text{-}1)!\ \left(\mathrm{Q}_{\mathrm{m}}(\nu)-Q_m(\nu+1)\right), \tag{1.3}$$

$$c_{\nu}^{(\mathrm{m})} = (\mathrm{m}\text{-}1)!\left(\mathrm{Q}_{\mathrm{m}\text{-}1}(\nu+1)-2Q_{m-1}(\nu)+Q_{m-1}(\nu-1)\right), \tag{1.4}$$

$$b_{\nu}^{(\mathrm{m})} = (\mathrm{m}\text{-}1)!\ \mathrm{Q}_{\mathrm{m}+1}(\nu+1), \tag{1.5}$$

Where,

$$Q_{m+1}(x)=\frac{1}{m!}\sum_{i=0}^{m+1}(-1)^{i}\binom{m+1}{i}(x-i)_{+}^{m}$$ is a B-Spline.

Theorem 2: Under the same condition of the theorem 1, there are the following relations:

$$\sum_{\nu=-1}^{m-1}\bar{a}_{\nu}^{(m)}\,s\left(\left(\nu+\frac{1}{2}\right)h\right)=h\sum_{\nu=-1}^{m-1}\bar{b}_{\nu}^{(m)}\,s'\left(\left(\nu+\frac{1}{2}\right)h\right), \tag{1.6}$$

$$\sum_{\nu=-1}^{m-1}\bar{c}_{\nu}^{(m)}\,s\left(\left(\nu+\frac{1}{2}\right)h\right)=h^{2}\sum_{\nu=-1}^{m-1}\bar{b}_{\nu}^{(m)}\,s''\left(\left(\nu+\frac{1}{2}\right)h\right), \tag{1.7}$$

Whose coefficient is:-

$$\bar{a}_{\nu}^{(m)}=(\mathrm{m\text{-}1})!\left(Q_{\mathrm{m}}\left(\nu+\frac{1}{2}\right)-Q_{m}\left(\nu+\frac{3}{2}\right)\right), \tag{1.8}$$

$$\bar{c}_{\nu}^{(m)}=(\mathrm{m\text{-}1})!\left(Q_{\mathrm{m\text{-}1}}\left(\nu+\frac{3}{2}\right)-2Q_{m-1}\left(\nu+\frac{1}{2}\right)+Q_{m-1}\left(\nu-\frac{1}{2}\right)\right), \tag{1.9}$$

$$\bar{b}_{\nu}^{(\mathrm{m})}=(\mathrm{m\text{-}1})!\;Q_{\mathrm{m+1}}\left(\nu+\frac{3}{2}\right). \tag{1.10}$$

We will consider the existence and uniqueness theorem for the approximate solution and we will check the convergence of the differential equation for $m=3$.

$-y''(x)+\sigma(x)(x)==f(x),\quad \sigma(\mathrm{x})\geq 0$, with boundary conditions
$y(0)=\alpha,\quad \mathrm{y(1)}=\beta$

Cubic Spline Interpolation (for $m=3$)

A cubic spline function $w(x)$ in interval $[x_0,\mathrm{x}_1]$, starting with point x_0 can be given as:

$$w(x)=a+b(x-x_0)+c(x-x_0)^2+d_0(x-x_0)^3 \tag{2.1}$$

As we proceed to next interval $[x_1,\mathrm{x}_2]$, we add a term $d_1(x-x_1)^3$ in the equation (2.1). Similarly for interval $[x_2,\mathrm{x}_3]$, we add $d_2(x-x_2)^3$, for interval $[x_3,\mathrm{x}_4]$,we add $d_3(x-x_3)^3$,..... for interval $[x_{n-1},\mathrm{x}_n]$,we add $d_{n-1}(x-x_{n-1})^3$, therefore spline function for intervals can be defined as:

$$w(x)=a+b(x-x_0)+c(x-x_0)^2+\sum_{i=0}^{n-1}d_i(x-x_i)_{+}^{3} \tag{2.2}$$

where,

$$\mathrm{x}_{+}^{\mathrm{m}}=\begin{cases}x^{m} & if\ \ \mathrm{x}>0\\ 0 & if\ \ \mathrm{x}\leq 0\end{cases}$$

We will obtain (n+3) coefficients from the equation (2.2). Out of (n+3) coefficient, two coefficients can be determine by the given boundary conditions and the remaining (n+1) coefficient are to be determined with the help of (n+1) nodes appeared in function. The satisfaction of the differential equation by these (n+1) nodes provides the required numbers of equations.
The equation to be satisfied at the i[th] node is:-

$$-w_i''+\sigma_i w_i=f_i \qquad 0\leq \mathrm{i}\leq \mathrm{n}. \tag{2.3}$$

In above equation we also add those parameter obtained by satisfying boundary conditions, namely,

$$w_0 = \alpha, \qquad w_n = \beta. \tag{2.4}$$

For cubic spline function $w(x)$ there are the following relationships from theorem 1.

$$w_{i+1} - 2w_i + w_{i-1} = \frac{h^2}{6}\left(w''_{i+1} + 4w''_i + w''_{i-1}\right), \quad 1 \le \text{i} \le \text{n-1} \tag{2.5}$$

$$w_{i+1} - w_{i-1} = \frac{h}{3}\left(w'_{i+1} + 4w'_i + w'_{i-1}\right), \quad 1 \le \text{i} \le \text{n-1} \tag{2.6}$$

We can also see a relationship between w'_v and w''_v , $v = i, \text{i+1}$

$$w'_{i+1} - w'_i = \frac{h}{2}\left(w''_{i+1} + w''_i\right), \tag{2.7}$$

From (2.3) and (2.5), we obtain the relation

$$\left(-1 + h^2\frac{\sigma_{i+1}}{6}\right)w_{i+1} + \left(2 + \frac{2}{3}h^2\sigma_i\right)w_i + \left(-1 + h^2\frac{\sigma_{i-1}}{6}\right)w_{i-1}$$

$$= \frac{h^2}{6}\left(f_{i+1} + 4f_i + f_{i-1}\right), \ 1 \le \text{i} \le \text{n-1} \tag{2.8}$$

If we write $W = \left(w_1, w_2, w_3 \ldots\ldots\ldots w_{n-1}\right)^T$, then the set of equations given by (2.8) may be written in the form

$$AW = K \tag{2.9}$$

Where K is a vector with (n-1) components,

$$k_1 = \frac{1}{6}\left(f_0 + 4f_1 + f_2\right) - \frac{\alpha}{h^2}\left(-1 + \frac{\sigma_0}{6}h^2\right),$$

$$k_i = \frac{1}{6}\left(f_{i-1} + 4f_i + f_{i+1}\right), \quad 2 \le \text{i} \le \text{n-2}$$

$$k_{n-1} = \frac{1}{6}\left(f_{n-2} + 4f_{n-1} + f_n\right) - \frac{\beta}{h^2}\left(-1 + \frac{\sigma_n}{6}h^2\right),$$

And the matrix A is a tri-diagonal matrix of order (n-1)

$$a_{i,\,\text{i-1}} = \frac{1}{h^2}\left(-1 + h^2\frac{\sigma_{i-1}}{6}\right), \quad a_{i,\,\text{i}} = \frac{1}{h^2}\left(2 + \frac{2h^2}{3}\sigma_i\right),$$

$$a_{i,\,\text{i+1}} = \frac{1}{h^2}\left(-1 + \frac{h^2}{6}\sigma_{i+1}\right)$$

As A is a tri-diagonal M-matrix $\left([4 : p.85]\right)$, where $0 < \text{h} \le \sqrt{\frac{6}{\max\,\sigma(\text{x})}}$, this matrix problem can be directly solved by a simple algorithm?

From (2.3), (2.6) and (2.8), we get the relations:-

$$w'_0 = w'_1 - \frac{h}{2}\left(\sigma_1 w_1 + \sigma_0 w_0 - f_0 - f_1\right), \tag{2.10}$$

$$w'_i = \frac{1}{2h}\left(w_{i+1} - w_{i-1}\right) - \frac{h}{12}\left(\sigma_{i+1}w_{i+1} - \sigma_{i-1}w_{i-1} - f_{i+1} + f_{i-1}\right),$$

$$1 \le \text{i} \le \text{n-1} \tag{2.11}$$

$$w'_n = w'_{n-1} + \frac{h}{2}\left(\sigma_n w_n + \sigma_{n-1}w_{n-1} - f_n - f_{n-1}\right). \tag{2.12}$$

From (2.10), (2.11) and (2.12), the derivatives at nodes can be calculated, and w(x) can be obtained by Hermite interpolation by a polynomial of degree 3 on each interval.

Theorem 3: If the solution y(x) of the equation is in $C^4[0,\ 1]$, then we have the following constants K_1 and K_2 such that:

$$|y(x) - w(x)| \le K_1 h^2, \qquad |y'(x) - w'(x)| \le K_2 h. \tag{2.13}$$

Proof. By Taylor expansions and Gerschgorin type convergence arguments, we can easily obtain

$$|y_i - w_i| \le Kh^2. \tag{2.14}$$

From (2.10), (2.11) and (2.12), we get

$$|y_i' - w_i'| \le Kh. \tag{2.15}$$

To complete the proof of above theorem we need one more important result from the following theorem:-

Theorem 4: Let $\Upsilon(x) \in C^t[0,1]$ *with* t ≥ 2m, and w(x) be in $\left(S_{2m-1},\ C^{m-1}\right)$ such that for some $p,\ (2m\text{-}2 \le p \le 2m)$,

$$\left|D^t\left((x_i) - w(x_i)\right)\right| \le Mh^{p-t} \quad \forall\ 0 \le i \le n,\ 0 \le t \le m-1.$$

If $x \in [0,\ 1]$, then

$$\left|D^k\left(\Upsilon(x) - w(x)\right)\right| \le Mh^{p-k} \quad \forall k \quad \text{with } 0 \le k \le m-1.$$

Proof. Let us consider the polynomial $p(x) \in \left(S_{2m-1},\ C^{m-1}\right)$ which interpolates the data, $\Upsilon^{(v)}(x_k),\ \Upsilon^{(v)}(x_{k+1}),\ v = 0,1,2,.....(m\text{-}1)$. then there exist a constant M_1 such that:-

$$\left|D^k\left(\Upsilon(x) - p(x)\right)\right| \le M_1 h^{2m-k} \quad \forall k \quad \text{with } 0 \le k \le m-1. \tag{2.16}$$

The proof of which is due to $[2 : p.408]$.

For estimating $p(x) - w(x)$ in each interval $I_k = [kh,\ (k+1)h]$, we use Hermite interpolation formula

$$p(x) - w(x) = \sum_{j=0}^{m-1} h^j \left\{ \begin{array}{l} \left(p^{(j)}(kh) - w^{(j)}(hk)\right)\lambda_j\left(\dfrac{x-kh}{h}\right) \\ +(-1)^j\left(p^{(j)}((k+1)h) - w^{(j)}((k+1)h)\right)\lambda_j\left(k+1-\dfrac{x}{h}\right) \end{array} \right\}, \tag{2.17}$$

Where $\lambda_j(t)$ are polynomials of degree (2m-1) which are the fundamental functions.

From the assumption of the theorem we have,

$$\left|p^{(j)}(kh) - w^{(j)}(hk)\right|, \left|p^{(j)}((k+1)h) - w^{(j)}((k+1)h)\right| \le Mh^{p-j}.$$

Therefore we have:

$$\left|D^k\left(p(x) - w(x)\right)\right| \le M_2 h^{p-k} \quad \forall k \quad \text{with } 0 \le k \le m-1. \tag{2.18}$$

On combining (2.16) and (2.18), we get the desired result.

Algorithm for C^3 – Cubic Spline function

The algorithm of cubic spline function can be discussed under following steps:

Step 1: Divide the interval [0, 1] into n small subintervals.

Step 2: We evaluate $S_0''' = y_0'''$ for given value $s(0) = y(0)$, $s'(0) = y'(0)$ and $s''(0) = y''(0)$

Step 3: $S_i'' = f(x_i, s_i)$ for i = 1, 2, 3, ...n, $x \in [x_{i-1}, x_i]$.

Step 4: Evaluate the value of s_i and s_i' for (n+1) equally spaced value, with the help of Taylor series expansion as:

$$s_i = s_{i-1} + hs_i' - \frac{h^2}{20}\left[3f\left(x_{i-1}, s_{i-1}\right) + 7f\left(x_i, s_i\right)\right] + \frac{h^3}{60}s''' \text{ and}$$

$$s_i' = s_{i-1}' + \frac{h}{2}\left[3f\left(x_{i-1}, s_{i-1}\right) + f\left(x_i, s_i\right)\right]$$

$$s_{i+1}''' = -5s_i''' + \frac{3}{h}\left[f\left(x_{i+1}, s_{i+1}\right) - f\left(x_{i-1}, s_{i-1}\right)\right]$$

Step 5: Evaluate:-

Step 6: If $i = k+1$, go to next step, otherwise repeat the process from step 4.

Step 7: Stop the process.

PROBLEM

Let us consider the problem:

$y'' = x + y, \quad y(0)=1, \; y'(0)=0, \quad x \in [0, 1]$

Results: Absolute maximum errors for s(x) and its derivatives with several values of 'h' is

h	$\lvert s_i - f_i \rvert$	$\lvert s_i - f_i \rvert'$	$\lvert s_i - f_i \rvert''$	$\lvert s_i - f_i \rvert'''$
$\frac{1}{10}$	3.14×10^{-6}	8.36×10^{-5}	1.27×10^{-4}	1.20×10^{-2}
$\frac{1}{20}$	2.01×10^{-7}	0.99×10^{-5}	3.70×10^{-10}	2.69×10^{-3}
$\frac{1}{100}$	3.21×10^{-10}	8.99×10^{-10}	8.40×10^{-11}	1.01×10^{-4}
$\frac{1}{1000}$	2.90×10^{-11}	4.49×10^{-9}	1.66×10^{-12}	2.26×10^{-5}

In above table $\lvert s_i - f_i \rvert$, $\lvert s_i - f_i \rvert'$, $\lvert s_i - f_i \rvert''$ and $\lvert s_i - f_i \rvert'''$ are absolute maximum errors in $s(x)$, s′(x), s″(x) and s‴(x) respectively.

Conflict of Interest: "The authors declare that they have no conflict of Interest".

CONCLUSION

To formulate the second order initial value problems, we have used cubic C^3 −Lacunary spline function in my work. The algorithm used in formulation of problem, gives better solution for second order initial value problems. In the formulation we have used small strips (h) to formulate and test the results. It is matter of research or experiments that the above algorithm will also provides the solution of higher order initial value problems.

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