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A Unified Fuzzy–Stochastic Framework for Performance Optimization in Queueing Systems

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Abstract

Classical stochastic queueing models rely on precisely specified arrival and service parameters, an assumption that is often difficult to justify in real service systems subject to measurement error, environmental variability, and managerial judgment. Such parameter ambiguity propagates nontrivially into performance measures, leading to policy prescriptions that may lack robustness. This work addresses this limitation by developing a unified fuzzy–stochastic framework in which key queueing parameters are modeled as fuzzy numbers embedded within Markovian service systems. Using an α–cut representation, the proposed formulation decomposes the hybrid model into a family of parametric stochastic queues, enabling analytical evaluation of performance measures including expected waiting time, mean queue length, and server utilization. These fuzzy performance descriptors are subsequently incorporated into a constrained optimization framework aimed at balancing congestionrelated costs and service capacity decisions. Structural properties of the resulting fuzzy performance measures are examined, and an α–level solution methodology is established. Numerical illustrations demonstrate the behavior of waiting time and stability regions under varying degrees of parameter imprecision. The applicability of the framework is illustrated through a service system motivated by customer-facing operations, such as banking or call center environments, highlighting the interpretability and practical relevance of the proposed approach.

Keywords: Queueing theory; Fuzzy–stochastic modeling; Performance optimization; Service systems; Operations research.

1 Introduction

Queueing systems constitute a fundamental modeling paradigm in operations research and applied probability, providing a mathematical representation for congestion, waiting, and service phenomena in a wide range of service and production environments. Applications arise naturally in banking systems, healthcare services, manufacturing lines, transportation networks, call centers, computer and communication systems, and inventory–service interfaces. In such settings, customers, jobs, packets, or tasks arrive dynamically, compete for limited service resources, and depart after completion of service. Queueing theory offers analytical tools to quantify performance measures such as mean waiting time, queue length, system utilization, and service delays, which are central to managerial decision-making and system design. Classical stochastic queueing models, including the well-known $M/M/1$ and $M/M/c$ systems, have therefore become indispensable in the analysis and optimization of service systems under uncertainty [8, 12, 13].

Despite their theoretical elegance and practical relevance, classical queueing models are built upon a set of idealized assumptions. Arrival and service processes are typically characterized by precise probability distributions with known parameters, such as constant arrival rates or exponential service times. In practice, however, these parameters are rarely known with certainty. Real-world service systems operate in environments influenced by demand fluctuations, human behavior, measurement errors, incomplete information, and managerial judgment. Arrival rates may vary due to seasonal effects or policy changes, while service rates may depend on worker fatigue, skill heterogeneity, or machine condition. Treating such inherently vague or imprecise quantities as fixed numerical constants may lead to misleading estimates of system performance and suboptimal decisions. This limitation has motivated researchers to seek modeling approaches that extend beyond purely stochastic descriptions based on precise parameter values [3, 5].

Fuzzy set theory provides a natural mathematical framework to represent imprecision and vagueness that cannot be adequately captured by classical probability theory alone. Introduced to formalize linguistic and subjective uncertainty, fuzzy sets allow parameters to be described by membership functions rather than exact numerical values. In the context of queueing systems, fuzzy arrival rates, fuzzy service rates, or fuzzy cost parameters can reflect expert assessments, managerial expectations, or incomplete data. Over the past decades, fuzzy queueing models have been proposed to incorporate such forms of uncertainty, enabling the computation of performance measures in terms of fuzzy numbers or intervals via α-cut representations [4, 14]. These models are particularly useful when historical data are insufficient for reliable statistical estimation or when uncertainty arises from qualitative factors rather than random variability.

Nevertheless, purely fuzzy queueing models also face limitations. Many service systems exhibit genuine stochastic variability that is well described by probabilistic mechanisms, such as random interarrival times or service completions driven by physical or human processes. Replacing stochastic parameters entirely by fuzzy quantities may obscure important probabilistic structures and lead to models that are difficult to validate empirically. As a result, neither classical stochastic models nor purely fuzzy formulations provide a complete representation of uncertainty in complex service systems. This observation has led to increasing interest in hybrid approaches that combine stochastic and fuzzy methodologies, allowing randomness and imprecision to coexist within a unified modeling framework [10, 15].

The integration of fuzzy and stochastic concepts in queueing theory is not merely a mathematical extension but a practical necessity for realistic system analysis and optimization. A unified fuzzy–stochastic framework enables the analyst to distinguish between aleatory uncertainty, modeled probabilistically, and epistemic uncertainty, modeled fuzzily. Within such a framework, random fluctuations in arrivals or services are treated through probability distributions, while uncertain or linguistically defined parameters are captured through fuzzy numbers. This hybrid perspective facilitates more robust performance evaluation and supports optimization decisions that remain meaningful under partial information. It also aligns well with contemporary decisionsupport systems in operations research, where data-driven analytics and expert judgment are jointly employed [4, 6].

In parallel with modeling advances, performance optimization remains a central objective in queueing analysis. Decision-makers are often interested not only in evaluating system behavior but also in determining optimal service capacities, staffing levels, or control policies that minimize waiting time, congestion cost, or operational risk. Incorporating fuzzy uncertainty into such optimization problems introduces additional analytical challenges, as objective functions and constraints may themselves become fuzzy-valued. Developing tractable optimization formulations that operate within a fuzzy–stochastic environment is therefore an important and timely research direction, particularly for service systems where uncertainty cannot be eliminated through data collection alone [2, 11].

The present study is motivated by the need for a coherent mathematical framework that unifies stochastic queueing theory and fuzzy uncertainty modeling, while remaining amenable to performance optimization and practical application. The focus is on developing a systematic approach in which classical queueing models are extended by fuzzy representations of uncertain parameters, without sacrificing the probabilistic structure essential for modeling random dynamics. By employing α-cut techniques and standard performance measures, the proposed framework allows analytical insights to be preserved while enhancing modeling flexibility. The emphasis is placed on interpretability, mathematical consistency, and applicability to real-world service systems.

The objectives and scope of this paper are as follows. The study aims to formulate a unified fuzzy–stochastic queueing framework that integrates probabilistic arrival and service mechanisms with fuzzy parameter representations. It seeks to derive performance measures within this framework and to embed them into an optimization model relevant to service-system design. A numerical illustration drawn from a representative service environment is used to demonstrate the applicability and interpretive value of the proposed approach. Through this development, the paper contributes to the literature by bridging the gap between stochastic queueing analysis and fuzzy decision modeling, providing a mathematically grounded tool for performance optimization under combined randomness and imprecision.

2 Preliminaries and Mathematical Background

This section presents the essential mathematical concepts and notation required for the development of the unified fuzzy–stochastic queueing framework. Standard results from queueing theory, probability, and fuzzy set theory are briefly recalled to ensure clarity and consistency in subsequent sections.

2.1 Basic Queueing Notation

A queueing system consists of customers (or jobs) arriving to receive service from one or more servers according to specified rules. Throughout this paper, standard Kendall's notation $A/B/c$ is adopted, where A denotes the interarrival time distribution, B the service time distribution, and c the number of parallel servers [8].

Let $\lambda > 0$ denote the mean arrival rate and $\mu > 0$ the mean service rate of a single server. For a system with c identical servers, the traffic intensity is defined as

$$\rho = \frac{\lambda}{c\mu},$$

and the stability condition requires $\rho < 1$. Standard performance measures include the expected number of customers in the system L, the expected number in the queue L_q, the mean sojourn time W, and the mean waiting time in the queue W_q. These measures are linked through Little's law,

$$L = \lambda W, \qquad L_q = \lambda W_q,$$

provided the system is stable and ergodic [5, 9].

2.2 Fuzzy Sets and Fuzzy Numbers

To represent imprecision in model parameters, fuzzy set theory is employed. Let X be a nonempty universal set. A fuzzy set $\tilde{A}$ in X is characterized by a membership function

$$\mu_{\tilde{A}} : X \to [0,1],$$

where $\mu_{\tilde{A}}(x)$ denotes the degree to which $x \in X$ belongs to $\tilde{A}$ [14].

A fuzzy number $\tilde{a}$ is a special fuzzy set on R that is normal, convex, and has a piecewise continuous membership function with bounded support [4]. Among various forms of fuzzy numbers, triangular fuzzy numbers are widely used due to their simplicity and interpretability. A triangular fuzzy number $\tilde{a} = (a_1, a_2, a_3)$ is defined by the membership function

□

( ) =

□□□□□□□□□0ax,2

$--aa11$, $\mu_{\tilde{a}}$ x $a3 - x$

$x \le a_1$, $a_1 \le$
□□□□□□□ $x \le a_2$, $a_2 \le$
□□0a,3 − $a2$ $x \le a_3$, $x \ge$
, a_3,

where $a_1 < a_2 < a_3$.

2.3 α-Cut Representation

A central tool for analyzing fuzzy numbers is the α-cut representation. For a fuzzy number $\tilde{a}$, the α-cut at level $\alpha \in [0,1]$ is defined as the crisp interval

$$[\tilde{a}]_\alpha = \{x \in \mathbb{R} : \mu_{\tilde{a}}(x) \geq \alpha\}.$$

For a triangular fuzzy number $\tilde{a} = (a_1, a_2, a_3)$, the α-cut takes the form

$$[\tilde{a}]_\alpha = [a_1 + \alpha(a_2 - a_1),\ a_3 - \alpha(a_3 - a_2)].$$

The α-cut approach allows fuzzy-valued performance measures to be analyzed through families of interval-valued problems, which is particularly convenient when integrating fuzzy parameters into stochastic queueing models [15].

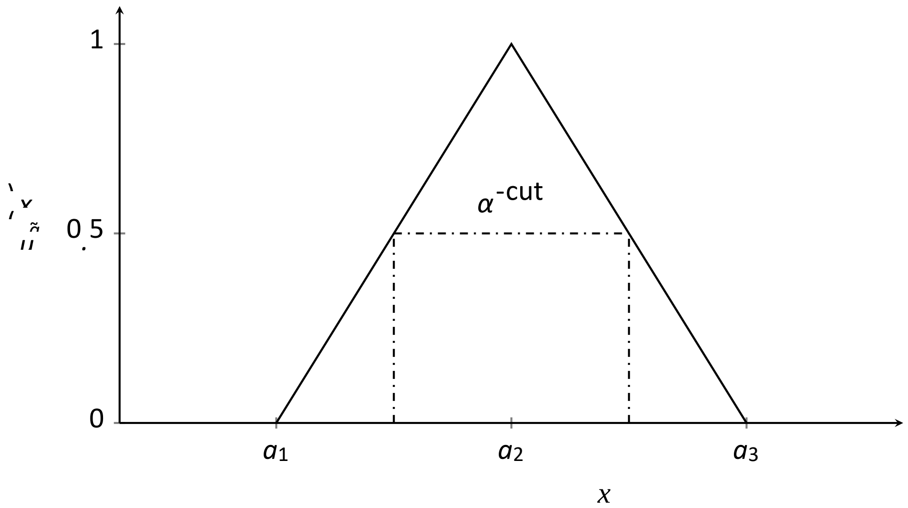

Figure 1: Triangular fuzzy number and its α-cut representation.

2.4 Probability Distributions Relevant to Queueing Systems

Queueing models rely heavily on probability distributions to describe random arrival and service processes. The Poisson distribution and its continuous-time counterpart, the Poisson process, are commonly used to model customer arrivals. If arrivals follow a Poisson process with rate λ, then the interarrival times are exponentially distributed with density

$$f(t) = \lambda e^{-\lambda t}, \qquad t \geq 0.$$

Similarly, exponential service times with rate μ are frequently assumed due to the memoryless property, which simplifies analytical treatment [12].

More general queueing systems may involve Erlang, hyperexponential, or general servicetime distributions, leading to $M/G/1$ or $G/G/c$ models. In the present study, exponential distributions are primarily considered to preserve analytical tractability while illustrating how fuzzy uncertainty can be incorporated into key system parameters. This probabilistic foundation, combined with fuzzy representations of imprecision, forms the basis for the unified modeling framework developed in subsequent sections [13].

3 Classical Queueing Model Formulation: $M/M/1$ and $M/M/c$

This section recalls the standard $M/M/1$ and $M/M/c$ queueing models that serve as the stochastic backbone for the fuzzy–stochastic extensions developed later. The presentation follows classical notation and results from queueing theory [5, 8, 12].

3.1 Model Assumptions and Parameters

Consider a service system with an infinite waiting room and the following stochastic primitives:

- **Arrival process:** customers arrive according to a Poisson process with rate $\lambda > 0$. Equivalently, interarrival times are i.i.d. exponential with mean $1/\lambda$.
- **Service mechanism:** service times are i.i.d. exponential with rate $\mu > 0$ (mean $1/\mu$).
- **Service discipline:** first-come, first-served (FCFS).

- **Number of servers:** either $c = 1$ ($M/M/1$) or $c \in \mathbb{N}$ ($M/M/c$), with identical servers operating in parallel.

Let $N(t)$ denote the number of customers in the system (queue plus service) at time t. Under the assumptions above, $\{N(t) : t \geq 0\}$ is a birth–death continuous-time Markov chain with birth rate λ and state-dependent death rate:

$$\delta_n = \begin{cases} n\mu, & \text{for } M/M/1 \text{ (since } n \in \{0,1\} \text{ in service)}, \\ \min(n,c)\mu, & \text{for } M/M/c. \end{cases}$$

In particular, for $M/M/1$, $\delta_n = \mu$ for all $n \geq 1$, whereas for $M/M/c$, $\delta_n = n\mu$ for $1 \leq n \leq c$ and $\delta_n = c\mu$ for $n \geq c$.

3.2 Traffic Intensity and Stability Condition

The *traffic intensity* (also called utilization factor) is defined by

$$\rho = \begin{cases} \dfrac{\lambda}{\mu}, & (M/M/1), \\ \dfrac{\lambda}{c\mu}, & (M/M/c). \end{cases}$$

A necessary and sufficient condition for the existence of a stationary distribution (positive recurrence) is the stability condition

$$\rho < 1.$$

This condition ensures that the long-run arrival workload is strictly less than the long-run service capacity [5, 12].

3.3 Performance Measures

The primary steady-state performance measures studied in this paper are:

- $L := \mathrm{E}[N]$, the expected number of customers in the system.
- $L_q := \mathrm{E}[N_q]$, the expected number of customers waiting in queue (excluding those in service).
- $W := \mathrm{E}[T]$, the expected time spent in the system (sojourn time).
- $W_q := \mathrm{E}[T_q]$, the expected waiting time in queue (excluding service time).

Whenever $\rho < 1$ and the usual regularity conditions hold, Little's relations apply:

$$L = \lambda W, \qquad L_q = \lambda W_q,$$

and

$$W = W_q + \frac{1}{\mu},$$

since the mean service time in both $M/M/1$ and $M/M/c$ is $1/\mu$ for an individual customer [5, 9].

3.4 *M/M/*1 Steady-State Formulas

For the $M/M/1$ model ($c = 1$), the stationary distribution is geometric:

$$\pi_n = (1-\rho)\rho^n, \qquad n = 0,1,2,\ldots,$$

where $\rho = \lambda/\mu < 1$. The standard performance measures are then

$$L = \frac{\rho}{1-\rho}, \qquad L_q = \frac{\rho^2}{1-\rho},$$

and, using Little's law,

$$W = \frac{1}{\mu - \lambda}, \qquad W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{\rho}{\mu - \lambda}.$$

These expressions quantify the rapid growth of congestion as $\rho \uparrow 1$.

3.5 *M/M/c* Steady-State Formulas

For the $M/M/c$ model with $c \geq 2$, define

$$a := \frac{\lambda}{\mu}, \quad \rho := \frac{\lambda}{c\mu} = \frac{a}{c}.$$

Let P_0 denote the steady-state probability of an empty system. Under $\rho < 1$,

$$P_0 = \left[\sum_{n=0}^{c-1} \frac{a^n}{n!} + \frac{a^c}{c!} \cdot \frac{1}{1-\rho}\right]^{-1}.$$

The probability that an arriving customer must wait (Erlang–C formula) is

$$P_{\text{wait}} = \frac{\frac{a^c}{c!} \cdot \frac{1}{1-\rho}}{\sum_{n=0}^{c-1} \frac{a^n}{n!} + \frac{a^c}{c!} \cdot \frac{1}{1-\rho}} = \frac{a^c}{c!} \cdot \frac{1}{1-\rho} P_0.$$

The mean queue length is

$$L_q = P_{\text{wait}} \frac{\rho}{1-\rho} = \frac{a^c}{c!} \cdot \frac{\rho}{(1-\rho)^2} P_0,$$

and hence

$$W_q = \frac{L_q}{\lambda}, \quad W = W_q + \frac{1}{\mu}, \quad L = L_q + \frac{\lambda}{\mu}.$$

These formulas reduce to the $M/M/1$ expressions when $c = 1$ and are standard in multiserver queue analysis [5, 12, 13].

Table 1: Steady-state performance measures for $M/M/1$ and $M/M/c$ (FCFS, infinite buffer), valid under $\rho < 1$.

Quantity	$M/M/1$ **($\rho = \lambda/\mu$)**	$M/M/c$ **($\rho = \lambda/(c\mu)$, $a = \lambda/\mu$)**
Stability	$\lambda < \mu$	$\lambda < c\mu$
P_0	$1 - \rho$	$\left[\sum_{n=0}^{c-1} \frac{a^n}{n!} + \frac{a^c}{c!} \frac{1}{1-\rho}\right]^{-1}$
Pwait	ρ	$\frac{a^c}{c!} \frac{1}{1-\rho} P_0$
L_q	$\frac{\rho^2}{1-\rho}$	$\frac{a^c}{c!} \frac{\rho}{(1-\rho)^2} P_0$
W_q	$\frac{L_q}{\lambda} = \frac{\lambda}{\mu(\mu-\lambda)}$	$\frac{L_q}{\lambda}$
W	$W_q + \frac{1}{\mu} = \frac{1}{\mu-\lambda}$	$W_q + \frac{1}{\mu}$
L	$\lambda W = \frac{\rho}{1-\rho}$	$L_q + \frac{\lambda}{\mu} = \lambda W$

4 Fuzzy–Stochastic Extension of the Classical Queueing Model

This section develops a fuzzy–stochastic extension of the classical $M/M/1$ and $M/M/c$ models by allowing key input parameters to be imprecise. Randomness in the arrival and service processes is retained through the Poisson/exponential assumptions, while epistemic uncertainty in the rates is modeled via fuzzy numbers. The resulting performance measures become fuzzy-valued and are computed through α-cut analysis [4, 10, 14, 15].

4.1 Fuzzy Parameterization of Arrival and Service Rates

In practice, the arrival rate and service rate are often estimated from limited data or expert judgment and may not be accurately represented by single crisp values. To capture such imprecision, we replace the classical parameters λ and μ by fuzzy numbers

$$\tilde{\lambda}, \qquad \tilde{\mu},$$

defined on R+ with membership functions $\mu_{\tilde{\lambda}}(\cdot)$ and $\mu_{\tilde{\mu}}(\cdot)$. A common choice in applications is the triangular fuzzy number representation $\tilde{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ and $\tilde{\mu} = (\mu_1, \mu_2, \mu_3)$, where the modal values λ_2, μ_2 correspond to the most plausible estimates and the spreads reflect uncertainty [4, 7].

For each $\alpha \in [0,1]$, define the α-cuts

$$[\tilde{\lambda}]_\alpha = [\lambda_L(\alpha), \lambda_U(\alpha)], \quad [\tilde{\mu}]_\alpha = [\mu_L(\alpha), \mu_U(\alpha)],$$

where $\lambda_L(\alpha) \leq \lambda_U(\alpha)$ and $\mu_L(\alpha) \leq \mu_U(\alpha)$. When $\tilde{\lambda}$ and $\tilde{\mu}$ are triangular, these intervals are given explicitly by

$$\lambda_L(\alpha) = \lambda_1 + \alpha(\lambda_2 - \lambda_1), \qquad \lambda_U(\alpha) = \lambda_3 - \alpha(\lambda_3 - \lambda_2),$$

$$\mu_L(\alpha) = \mu_1 + \alpha(\mu_2 - \mu_1), \qquad \mu_U(\alpha) = \mu_3 - \alpha(\mu_3 - \mu_2).$$

4.2 Fuzzy Stability Condition

The classical stability condition $\rho < 1$ must be reinterpreted because ρ becomes fuzzy when λ and μ are fuzzy. For $M/M/1$, define the fuzzy traffic intensity

$$\tilde{\rho} = \frac{\tilde{\lambda}}{\tilde{\mu}},$$

and for $M/M/c$, $\tilde{\rho} = \frac{\tilde{\lambda}}{c\tilde{\mu}}$.

Using α-cuts, the α-cut of $\tilde{\rho}$ is the interval

$$[\tilde{\rho}]_\alpha = \left[\frac{\lambda_L(\alpha)}{\mu_U(\alpha)\, c^{\mathbb{I}(c\geq 2)}}, \frac{\lambda_U(\alpha)}{c^{\mathbb{I}(c\geq 2)}\,\mu_L(\alpha)}\right],$$

where $\mathrm{I}(c \geq 2)$ is the indicator (equal to 1 for $M/M/c$, $c \geq 2$, and 0 for $M/M/1$). In particular:

$$[\tilde{\rho}]_\alpha(M/M/1) = \left[\frac{\lambda_L(\alpha)}{\mu_U(\alpha)}, \frac{\lambda_U(\alpha)}{\mu_L(\alpha)}\right], \qquad [\tilde{\rho}]_\alpha(M/M/c) = \left[\frac{\lambda_L(\alpha)}{c\mu_U(\alpha)}, \frac{\lambda_U(\alpha)}{c\mu_L(\alpha)}\right].$$

A practical and mathematically consistent feasibility requirement is that the system be stable for all realizations within each α-cut, namely

$$\sup[\tilde{\rho}]_\alpha < 1 \qquad \text{for the relevant range of } \alpha.$$

Equivalently, for each α,

$$\frac{\lambda_U(\alpha)}{\mu_L(\alpha)} < 1 \quad (M/M/1), \qquad \frac{\lambda_U(\alpha)}{c\mu_L(\alpha)} < 1 \quad (M/M/c).$$

(α) $c\mu$

This condition ensures that the classical steady-state formulas remain valid under any crisp selection of (λ,μ) within the α-cut, preserving stochastic consistency [10, 15].

4.3 α-Cut Derivation of Fuzzy Performance Measures

Let $g(\lambda,\mu)$ denote a classical performance measure (such as L, L_q, W, or W_q) for the $M/M/1$ or $M/M/c$ model. The fuzzy counterpart is defined through Zadeh's extension principle [14], but for computation and transparency we adopt the equivalent α-cut method:

$$[\tilde{g}]_\alpha = g_L(\alpha),\ g_U(\alpha),$$

where

$$g_L(\alpha) = \min\{g(\lambda,\mu) : \lambda \in [\tilde{\lambda}]_\alpha,\ \mu \in [\tilde{\mu}]_\alpha,\ \text{stability holds}\},\ g_U(\alpha) = \max\{g(\lambda,\mu) : \lambda \in$$

$$[\tilde{\lambda}]_\alpha,\ \mu \in [\tilde{\mu}]_\alpha,\ \text{stability holds}\}.$$

In the present setting, the relevant performance measures are monotone in λ and μ on the stable region: for fixed μ, they increase with λ; for fixed λ, they decrease with μ. This monotonicity yields closed-form α-cut bounds by evaluating g at the interval endpoints [4, 7].

4.3.1 Fuzzy performance measures for $M/M/1$

For $M/M/1$, the classical expressions are

$$W(\lambda,\mu) = \frac{1}{\mu-\lambda}, \qquad W_q(\lambda,\mu) = \frac{\lambda}{\mu(\mu-\lambda)},$$

$$L(\lambda,\mu) = \frac{\lambda}{\mu-\lambda}, \qquad L_q(\lambda,\mu) = \frac{\lambda^2}{\mu(\mu-\lambda)}.$$

Fix $\alpha \in [0,1]$ and let $\lambda \in [\lambda_L(\alpha),\lambda_U(\alpha)]$, $\mu \in [\mu_L(\alpha),\mu_U(\alpha)]$, with $\lambda < \mu$. Since $\mu - \lambda$ appears in the denominator, the largest congestion occurs when λ is large and μ is small. Hence,

$$[\tilde{W}]_\alpha = \mu_U(\alpha) - 1\ \lambda_L(\alpha),\ \mu\ \frac{1}{\alpha\ -\lambda_U(\alpha)} L(\ ),$$

provided $\mu_L(\alpha) > \lambda_U(\alpha)$. Similarly,

$$[\tilde{L}]_\alpha = \mu_U(\alpha\lambda)L - (\alpha\lambda)L(\alpha),\ \mu L(\alpha\lambda)U - (\alpha\lambda)U(\alpha).$$

For the queue-only measures,

$$[\tilde{W}_q]_\alpha = \left[\frac{\lambda_L(\alpha)}{\mu_U(\alpha)\,(\mu_U(\alpha) - \lambda_L(\alpha))},\ \frac{\lambda_U(\alpha)}{\mu_L(\alpha)\,(\mu_L(\alpha) - \lambda_U(\alpha))}\right],$$

$$[\tilde{L}_q]_\alpha = \left[\frac{\lambda_L(\alpha)^2}{\mu_U(\alpha)\,(\mu_U(\alpha) - \lambda_L(\alpha))},\ \frac{\lambda_U(\alpha)^2}{\mu_L(\alpha)\,(\mu_L(\alpha) - \lambda_U(\alpha))}\right].$$

These α-cut intervals define fuzzy-valued performance measures $\tilde{W}$, $\tilde{W}_q$, $\tilde{L}$, and $\tilde{L}_q$.

4.3.2 Fuzzy performance measures for *M/M/c* For $M/M/c$, performance measures depend on λ and μ through

$$\lambda \quad \lambda\ a = \_, \rho = \_\_. \mu \qquad c\mu$$

The classical steady-state quantities are:

$$a$$

$$P(\lambda,\mu) = "cX-1\ nn!\ \ acc \qquad \underline{1}\#-1$$

$$P\text{wait}(\lambda,\mu) = \cancel{a}cc!\ 1\ -\underline{1\ }\rho\ \ P^1,\quad L_q(\lambda,\mu) = {}^{P}1^{\text{wait}}{}_{-\rho}{}^{\rho}, \quad W_q(\lambda,\mu) = {}^{L}\_\_{}_{\lambda}{}^{q}, \quad W(\lambda,\mu) = W_q + {}_{\mu}\underline{1}, \quad L(\lambda,\mu) = L_q + {}_{\mu}\_{}^{\lambda}.$$

To compute fuzzy counterparts, define for each α:

$$[\tilde{\lambda}]_\alpha = [\lambda_L(\alpha), \lambda_U(\alpha)], \qquad [\tilde{\mu}]_\alpha = [\mu_L(\alpha), \mu_U(\alpha)],$$

and enforce the stable region

$$\lambda < c\mu.$$

Because the mapping $(\lambda,\mu)\ 7\!\!\longrightarrow (L_q, W_q, W, L)$ is increasing in λ and decreasing in μ within the stable region, the α-cut bounds are obtained by endpoint evaluation:

$$[\tilde{L}_q]_\alpha = L_q\, \lambda_L(\alpha), \mu_U(\alpha),\ L_q\, \lambda_U(\alpha), \mu_L(\alpha),$$

$$[\tilde{W}\ q\ ]\alpha = "Lq\ \lambda L\lambda\cancel{(\alpha L()\alpha,\mu)U(\alpha),\ Lq}\ \lambda U\cancel{\lambda(U\alpha()\alpha,\mu)L(\alpha)\#,}\ [\tilde{W}\ ]\alpha = "L^q$$

$$\cancel{\lambda^L\lambda(\alpha L()_\alpha,\mu)^U(\alpha) + {}_\mu}U1\cancel{(_\alpha),\ L^q\ \lambda^U{}_\lambda(U\alpha()_\alpha,\mu)^L(\alpha) + {}_\mu L1}(_\alpha)^{\#},$$

$$[\tilde{L}\ ]\alpha = Lq\ \lambda L(\alpha),\mu U(\alpha) + \mu\lambda\cancel{L((\alpha\alpha)}),\ Lq\ \lambda U(\alpha),\mu L(\alpha) + \lambda\mu U\cancel{L((\alpha\alpha)}).$$

$$U$$

These formulas define fuzzy performance measures through α-cuts without altering the underlying stochastic dynamics of arrivals and services. For numerical implementation, one evaluates P_0, P_{wait}, and L_q at the relevant endpoints for each α. Such interval-based computations are standard in fuzzy queueing analysis and provide transparent bounds on system performance [1, 10, 15].

4.4 Interpretation of Fuzzy Waiting Time and Queue Length

In the fuzzy–stochastic framework, $\tilde{W}$, $\tilde{W}_q$, $\tilde{L}$, and $\tilde{L}_q$ are fuzzy numbers (or fuzzy intervals) representing a family of plausible performance levels induced by imprecision in λ and μ. The membership value associated with a candidate performance level reflects its compatibility with the imprecise input information. The α-cut $[\tilde{W}]_\alpha$, for example, is an interval of waiting times that are supported at confidence level α by the fuzzy parameter descriptions. Larger α corresponds to more conservative, high-belief regions near the modal parameter values, yielding narrower performance intervals. Smaller α admits broader parameter uncertainty and thus wider performance ranges [4, 14].

Operationally, fuzzy queue length $\tilde{L}$ provides a quantified band for expected congestion, while fuzzy waiting time $\tilde{W}$ provides a quantified band for expected delay. These bands are particularly useful in capacity planning and service-level design: one may, for instance, seek server levels c such that the upper bound of $[\tilde{W}_q]_\alpha$ remains below a prescribed service target for a chosen α. Hence, the fuzzy–stochastic measures support decision-making under combined randomness (in arrivals and services) and imprecision (in rate estimation), without forcing a reduction of uncertainty to a single scalar estimate.

[1] $n=0 \qquad + \qquad !\ 1-\rho \quad ,$

The stepwise α-cut derivations above preserve mathematical consistency by ensuring that for each α, the performance bounds are computed over parameter intervals restricted to the stable region. This approach yields nested intervals across α (monotonicity in α) and recovers the classical stochastic measures when $\tilde{\lambda}$ and $\tilde{\mu}$ degenerate to crisp numbers. Such properties are essential for a coherent integration of fuzzy uncertainty with stochastic queueing theory [10, 15].

5 Optimization Model for Fuzzy–Stochastic Queue Performance

This section formulates an optimization problem in which service capacity decisions are selected to improve system performance under combined stochastic variability and fuzzy parameter uncertainty. The stochastic structure of the queue is retained through the $M/M/c$ model, while the arrival and service rates are represented by fuzzy numbers $\tilde{\lambda}$ and $\tilde{\mu}$. The decision variable is the number of parallel servers $c \in \mathbb{N}$, although the formulation can be extended to include controllable service rates or staffing levels.

5.1 Assumptions

The optimization model is developed under the following assumptions:

1. Arrivals follow a Poisson process with rate λ, and service times are i.i.d. exponential with rate μ per server, consistent with the $M/M/c$ setting.
2. Parameter imprecision is modeled by fuzzy numbers $\tilde{\lambda}$ and $\tilde{\mu}$ on $\mathbb{R}^+$. For each $\alpha \in [0,1]$, the α-cuts are

   $$[\tilde{\lambda}]_\alpha = [\lambda_L(\alpha), \lambda_U(\alpha)], \qquad [\tilde{\mu}]_\alpha = [\mu_L(\alpha), \mu_U(\alpha)].$$

3. Service discipline is FCFS, with an infinite waiting room and no customer abandonment.
4. The system must be stable at a specified confidence level $\alpha_0 \in (0,1]$, reflecting the planner's desired degree of conservatism.
5. Costs consist of (i) staffing/service capacity cost proportional to the number of servers and (ii) waiting-related cost proportional to expected delay or queue length.

5.2 Decision Variable and Performance Metric

Let $c \in \mathbb{N}$ denote the number of servers. Under fuzzy parameters, the mean waiting time in queue becomes a fuzzy quantity $\tilde{W}_q(c)$. For each $\alpha \in [0,1]$, its α-cut is

$$[\tilde{W}_q(c)]_\alpha = W_{q,L}(c;\alpha),\ W_{q,U}(c;\alpha),$$

where, by monotonicity of W_q in λ and μ on the stable region,

$$W_{q,L}(c;\alpha) = W_q(\lambda_L(\alpha), \mu_U(\alpha), c), \qquad W_{q,U}(c;\alpha) = W_q(\lambda_U(\alpha), \mu_L(\alpha), c),$$

and $W_q(\lambda,\mu,c)$ is the classical $M/M/c$ waiting time in queue:

$$W_q(\lambda,\mu,c) = \frac{L_q(\lambda,\mu,c)}{\lambda}, \qquad L_q(\lambda,\mu,c) = P_{\text{wait}}(\lambda,\mu,c)\frac{\rho}{1-\rho}, \quad \rho = \frac{\lambda}{c\mu}.$$

5.3 Objective Function

Two standard objective specifications are useful in practice.

(A) Minimization of a conservative expected waiting time (service-level driven) Fix a confidence level $\alpha_0 \in (0,1]$. A conservative performance objective is to minimize the upper bound of the α_0-cut:

$$\min_{c \in \mathbb{N}} W_{q,U}(c;\alpha_0).$$

This criterion ensures that waiting time is controlled even under unfavorable (high arrival, low service) parameter realizations supported at level α_0.

(B) Minimization of a fuzzy–stochastic total cost (economic trade-off)

Let $h > 0$ denote the unit waiting cost per customer per unit time and $s > 0$ denote the cost per server per unit time. A classical proxy for expected waiting cost is hL_q (by Little's law, $L_q = \lambda W_q$). Under fuzzy parameters, define a conservative total cost at level α_0:

$$C_U(c;\alpha_0) = sc + hL_{q,U}(c;\alpha_0),$$

where

$$L_{q,U}(c;\alpha_0) = L_q(\lambda_U(\alpha_0),\mu_L(\alpha_0),c).$$

The optimization problem becomes

$$\min_{c\in\mathbb{N}} C_U(c;\alpha_0).$$

More generally, one may combine delay and capacity costs through weighted objectives, or use alternative fuzzy ranking indices (centroid, signed distance, possibility/necessity measures) to map $\tilde{C}(c)$ to a scalar criterion [4, 10, 15].

5.4 Constraints

Stability (feasibility) constraint

To preserve the validity of steady-state $M/M/c$ measures, the system must be stable. A conservative α_0-level stability condition is

$$\frac{\lambda_U(\alpha_0)}{c\mu_L(\alpha_0)} < 1.$$

Equivalently,

$$c > \frac{\lambda_U(\alpha_0)}{\mu_L(\alpha_0)}.$$

This constraint guarantees stability for all $(\lambda,\mu) \in [\tilde{\lambda}]_{\alpha 0} \times [\tilde{\mu}]_{\alpha 0}$ under the worst-case rate combination.

Service capacity bounds

Practical considerations impose lower and upper bounds on the number of servers:

$$c_{\min} \leq c \leq c_{\max}, \qquad c \in \mathbb{N},$$

where $c_{\min} \geq 1$ and $c_{\max}$ reflects space, workforce, or equipment limits.

Budget or cost-limit constraint (optional) If a fixed operating budget B is given, one may enforce

$$sc \leq B,$$

or, more conservatively, require the total cost upper bound not to exceed B:

$$C_U(c;\alpha_0) \leq B.$$

Such constraints are natural in staffing problems and service design.

5.5 Complete Optimization Formulation

A representative fuzzy–stochastic server-sizing model based on conservative total cost is:

$$\min_{c\in\mathbb{N}} \quad C_U(c;\alpha_0) = sc + hL_q(\lambda_U(\alpha_0),\mu_L(\alpha_0),c)$$

s.t. $c\mu\lambda U_L((\alpha\alpha 00)) < 1,$

$$c\min \le c \le c\max.$$

An analogous formulation is obtained by replacing $C_U(c;\alpha_0)$ with $W_{q,U}(c;\alpha_0)$ when the design objective is service-level compliance.

5.6 Solution Strategy

Since c is integer-valued and typically small-to-moderate in service applications, an efficient and transparent solution strategy is discrete search.

Analytical structure and monotonicity

For fixed (λ,μ) with $\rho < 1$, $W_q(\lambda,\mu,c)$ is decreasing in c, while the staffing cost term sc is increasing in c. Hence, the total cost objective often exhibits a unimodal pattern (decreasing initially due to congestion reduction, then increasing due to staffing cost), implying that the optimal $c^\star$ is attained within a compact feasible set and can be identified by evaluating successive values of c [5, 12].

Algorithmic procedure (recommended for implementation)

Fix α_0, compute $\lambda_U(\alpha_0)$ and $\mu_L(\alpha_0)$, and determine the smallest stable server count:

$$c0 = \max c\min,\ \lambda\mu U\cancel{L((\alpha\alpha 0}0)) + 1.$$

Then:

1. For each integer $c \in \{c_0, c_0 + 1, \ldots, c_{\max}\}$, compute P_0, P_{wait}, L_q, W_q using the classical $M/M/c$ formulas evaluated at $\lambda_U(\alpha_0), \mu_L(\alpha_0))$.
2. Evaluate the chosen objective (e.g., $C_U(c;\alpha_0)$ or $W_{q,U}(c;\alpha_0)$).
3. Select $c^\star$ as the minimizer over the feasible set.

If a less conservative decision rule is desired, one may compute both endpoints of the α_0-cut and adopt a fuzzy ranking approach, for example minimizing the midpoint

$$\frac{Wq,L(c;\alpha 0) + Wq,U(c;\alpha 0)}{2},$$

or minimizing a weighted combination of bounds. Such variants remain computationally straightforward and retain interpretability [4, 10, 15].

Extension to multi-objective design

When both waiting and staffing costs must be balanced under multiple service constraints, the model may be cast as a multi-objective problem with fuzzy goals (e.g., "keep W_q below a target" while controlling cost). Standard fuzzy goal programming or fuzzy constrained optimization techniques can then be employed [11, 15].

6 Numerical Case Study: Teller Staffing in a Bank Branch

This section presents a numerical case study motivated by a mid-sized bank branch during peak hours. Customers arrive randomly, join a single common queue, and are served by c parallel tellers under FCFS discipline. The branch manager seeks a staffing level c that achieves acceptable waiting performance under (i) inherent stochastic variability and (ii) imprecision in the estimated arrival and service rates. The stochastic backbone is an $M/M/c$ model [5, 12], while rate imprecision is described using fuzzy numbers and evaluated through α-cuts [1, 4, 10, 14, 15].

6.1 Parameter Values and Fuzzy Representations

Based on transaction logs and expert assessment, the branch estimates a modal peak arrival rate close to 30 customers/hour, with plausible variation due to day-to-day demand. Similarly, an individual teller completes a transaction in about 5 minutes on average, with variability across tellers and transaction types. These are modeled as triangular fuzzy numbers: $\tilde{\lambda} = (\lambda_1, \lambda_2, \lambda_3) = (28, 30, 33)$ customers/hour, $\tilde{\mu} = (\mu_1, \mu_2, \mu_3) = (11, 12, 13)$ customers/hour per teller.

For a chosen confidence level $\alpha_0 = 0.5$, the α-cuts are

$$[\tilde{\lambda}]_{0.5} = [\lambda_L, \lambda_U] = 29, 31.5, [\tilde{\mu}]_{0.5} = [\mu_L, \mu_U] = 11.5, 12.5.$$

In the fuzzy–stochastic setting, conservative (worst-case) waiting performance is obtained by evaluating $M/M/c$ formulas at $(\lambda, \mu) = (\lambda_U, \mu_L)$, while optimistic (best-case) performance is obtained at $(\lambda, \mu) = (\lambda_L, \mu_U)$. This endpoint evaluation is justified by monotonicity of W_q and L_q with respect to λ and μ on the stable region [4, 10, 15].

6.2 Computed Fuzzy–Stochastic Performance Measures

For each $c \in \{3, 4, 5, 6, 7\}$, stability is checked using the α_0-level conservative condition:

$$\rho_U(c; \alpha_0) = \frac{\lambda_U}{c\mu_L} < 1.$$

The classical $M/M/c$ formulas (Erlang–C) are then used to compute L_q, W_q, W, and L at the two endpoint pairs (λ_L, μ_U) and (λ_U, μ_L) [5, 12]. The resulting α_0-cut intervals $[\tilde{W}_q(c)]_{\alpha_0}$ and $[\tilde{L}_q(c)]_{\alpha_0}$ summarize performance uncertainty induced by imprecise rates.

Table 2: Fuzzy–stochastic performance at $\alpha_0 = 0.5$ for the bank branch ($\tilde{\lambda} = (28, 30, 33)$, $\tilde{\mu} = (11, 12, 13)$). Intervals report best-case (λ_L, μ_U) to worst-case (λ_U, μ_L) values.

c	ρ **interval**	W_q **interval (min)**	L_q **interval (cust.)**
3	[0.773, 0.913]	[4.266, 16.804]	[2.062, 8.822]
4	[0.580, 0.685]	[0.749, 1.677]	[0.362, 0.881]
5	[0.464, 0.548]	[0.181, 0.407]	[0.087, 0.214]
6	[0.387, 0.457]	[0.045, 0.111]	[0.022, 0.058]
7	[0.331, 0.391]	[0.011, 0.030]	[0.005, 0.016]

6.3 Figure: Waiting Time versus Number of Tellers

Figure 2 shows how the queueing delay decreases as the number of tellers increases, displayed as an α_0-cut band for W_q. The conservative curve uses (λ_U, μ_L) and the optimistic curve uses (λ_L, μ_U).

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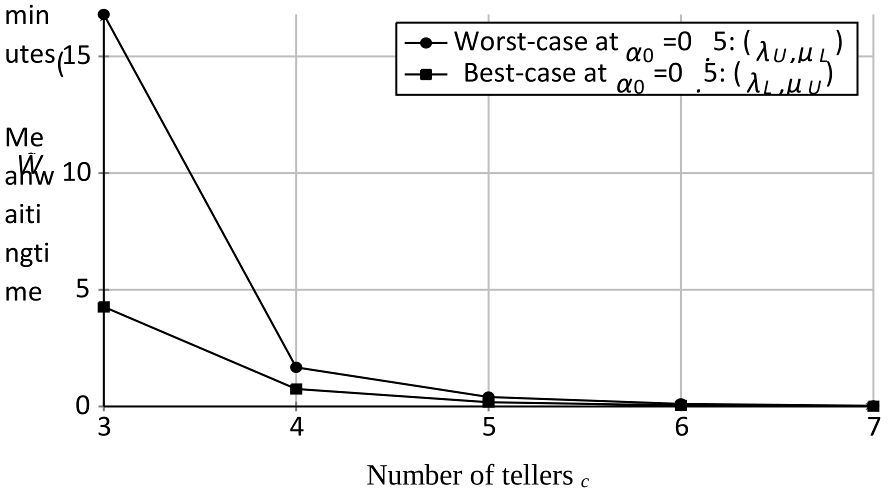

Figure 2: Waiting time in queue W_q versus number of tellers c at $\alpha_0 = 0.5$, forming an α-cut band between best-case and worst-case endpoints.

6.4 Interpretation: Mathematical and Operational

Mathematically, the results exhibit the expected nonlinear sensitivity of $M/M/c$ delays to utilization. With $c = 3$, the conservative utilization is $\rho_U \approx 0.913$, placing the system near the heavy-traffic regime; the resulting delay is large, with W_q ranging from about 4.27 to 16.80 minutes at $\alpha_0 = 0.5$. Increasing to $c = 4$ reduces utilization to at most $\rho_U \approx 0.685$, producing a sharp reduction in waiting time: $W_q \in [0.75, 1.68]$ minutes. Subsequent increments $c = 5, 6, 7$ continue to reduce delay, but the marginal gains diminish rapidly; the conservative W_q drops below half a minute already at $c = 5$, and below 0.12 minutes at $c = 6$. This diminishing-returns pattern is consistent with Erlang–C behavior when moving away from the high-utilization region [5, 12].

Operationally, the α_0-cut intervals quantify a performance band rather than a single point estimate. If the branch targets a conservative waiting time below 2 minutes during peak hours at $\alpha_0 = 0.5$, then $c = 4$ is the smallest staffing level meeting that target under the stated fuzzy uncertainty. If a stricter target is adopted (for example, sub-minute waiting at the same α_0), then $c = 5$ becomes appropriate. The table also indicates that expected queue length becomes negligible beyond $c = 5$ even in the conservative regime, suggesting that additional tellers primarily reduce already-small delays, which may be difficult to justify unless servicelevel agreements or reputational considerations require it. These observations motivate the cost-based optimization model in the next section, where staffing cost is explicitly traded against fuzzy–stochastic congestion cost [10, 15].

7 Discussion

The fuzzy–stochastic framework developed in this study provides a structured way to assess how imprecision in system parameters propagates into queueing performance measures. When arrival and service rates are modeled as fuzzy numbers rather than fixed constants, the resulting waiting times and queue lengths become fuzzy-valued quantities characterized by nested α-cut intervals.

Mathematically, this transformation replaces point estimates of W, W_q, L, and L_q with families of intervals whose widths depend on both the degree of fuzziness in the input parameters and the nonlinear structure of the queueing formulas. The case study demonstrates that these intervals expand rapidly as utilization approaches the stability boundary, reflecting the sensitivity of the underlying rational functions to perturbations in λ and μ. Hence, fuzziness amplifies uncertainty precisely in those operating regimes where congestion effects are most pronounced.

A key distinction from classical stochastic queueing analysis lies in the interpretation of performance measures. In the classical $M/M/c$ model with crisp parameters, quantities such as W_q and L_q are single-valued expectations, implicitly assuming that the arrival and service rates are known accurately. In contrast, the fuzzy–stochastic model recognizes that such rates may only be known within ranges supported by expert judgment or limited data. The α-cut–based measures obtained here preserve the stochastic structure of the queue while explicitly accounting for epistemic uncertainty. Importantly, when the fuzzy parameters collapse to degenerate (crisp) values, the interval-valued measures reduce to the classical results, ensuring consistency with standard queueing theory. Thus, the proposed framework can be viewed as a mathematically coherent generalization rather than a replacement of classical stochastic models.

From a mathematical perspective, the monotonicity of standard performance measures in λ and μ plays a central role in the tractability of the fuzzy–stochastic analysis. Because waiting time and queue length increase with arrival intensity and decrease with service capacity on the stable region, the extrema of each α-cut are attained at the interval endpoints. This property allows the use of closed-form expressions evaluated at worst-case and best-case parameter combinations, avoiding the need for complex nonlinear programming at each α-level. At the same time, the nonlinear dependence of W_q and L_q on the utilization factor explains why the width of the fuzzy performance intervals is highly asymmetric near $\rho \approx 1$, a phenomenon that cannot be captured by linear sensitivity analysis alone.

The managerial and operational implications of these findings are significant. Interval-valued performance measures provide decision-makers with explicit bounds on expected congestion rather than optimistic point forecasts. For instance, in staffing problems, a manager can choose a server level c that guarantees an upper bound on waiting time below a prescribed service threshold at a selected confidence level α. This is fundamentally different from classical optimization based on mean values, where small underestimation of arrival rates may lead to severe congestion. The fuzzy–stochastic approach thus supports more robust capacity planning, especially in environments where demand forecasts are imprecise or rapidly changing.

Furthermore, the results highlight the economic relevance of operating away from highutilization regimes. The numerical study shows that modest increases in service capacity can sharply reduce both the level and the uncertainty of waiting time when the system is near saturation. Once utilization is sufficiently low, further capacity additions yield diminishing marginal improvements in performance and narrow fuzzy intervals only slightly. This observation provides a quantitative basis for balancing service quality and cost under uncertainty, reinforcing the role of fuzzy–stochastic analysis as a decision-support tool rather than a purely descriptive model.

Incorporating fuzziness into stochastic queueing models does not merely add uncertainty to performance measures; it reshapes how congestion risk is quantified and managed. By embedding imprecise parameters within a rigorous stochastic framework, the proposed approach delivers mathematically interpretable performance bands and actionable insights for service-system design, while remaining firmly grounded in classical queueing theory.

8 Conclusion

This paper has presented a unified fuzzy–stochastic framework for analyzing and optimizing queueing systems in the presence of both random variability and parameter imprecision. By integrating classical $M/M/1$ and $M/M/c$ queueing models with fuzzy representations of arrival and service rates, the framework preserves the probabilistic structure of queue dynamics while explicitly accounting for epistemic uncertainty. The use of α-cut analysis provides a transparent and mathematically consistent mechanism for deriving fuzzy-valued performance measures, ensuring that classical stochastic results are recovered as special cases when uncertainty collapses to crisp parameters.

From an analytical standpoint, the study demonstrates that standard queueing performance measures exhibit monotone dependence on imprecise parameters within the stability region, enabling closed-form interval bounds for waiting time and queue length. The results reveal how fuzziness amplifies uncertainty in congested regimes and how this effect diminishes as service capacity increases. The optimization model formulated on the basis of conservative α-level performance measures further illustrates how staffing and capacity decisions can be structured to guarantee stability and acceptable service performance under worst-case realizations supported by the fuzzy parameter descriptions.

The practical relevance of the proposed framework is illustrated through a numerical case study motivated by a real-world banking service system. The fuzzy–stochastic analysis yields performance bands rather than single-point estimates, allowing managers to evaluate congestion risk explicitly and to select service capacities that balance delay reduction against operational cost. Such interval-based performance information is particularly valuable in environments characterized by limited data, demand volatility, or reliance on expert judgment, where classical point-estimate models may provide a misleading sense of precision.

Several extensions of the present work offer promising directions for future research. The framework may be generalized to interconnected multi-server networks, where imprecision propagates through routing and feedback effects. Game-theoretic queue control models could be developed to study strategic interactions among service providers or customers under fuzzy–stochastic uncertainty. In addition, robust or risk-sensitive optimization criteria, incorporating possibility, necessity, or downside-risk measures, could be integrated to further enhance decision-making under uncertainty. These extensions would broaden the applicability of the fuzzy–stochastic approach while maintaining its mathematical rigor and operational relevance.

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