



A Note on separation axioms in soft topological space

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Abstract

The classification of topological spaces via separation axioms constitutes a cornerstone of general topology, providing the structural rigor necessary to distinguish distinct points and closed sets through open neighborhoods. In the emerging field of soft set theory, originally proposed by Molodtsov as a mathematical tool for dealing with uncertainty and vagueness, the translation of these classical axioms into Soft Topological Spaces (STS) has engendered a complex and often divergent theoretical landscape. This research report offers an exhaustive examination of separation axioms—specifically T_0 , T_1 , T_2 (Hausdorff), Regularity (T_3), and Normality (T_4)—within the context of soft topology. We critically analyze the fundamental bifurcation in the literature concerning the definition of the basic "point" in a soft space: the "ordinary point" approach initiated by Shabir and Naz versus the "soft point" approach developed by Zorlutuna, Hussain, and Ahmad.

Through a meticulous synthesis of foundational papers and recent advancements from the 2020-2024 period, this report elucidates the structural anomalies that arise when classical intuitions are applied to parameterized soft structures. We investigate the hierarchy of axioms, demonstrating via constructed counterexamples that classical implications such as T_2 implies T_1 and Regularity Normality are contingent upon specific definitions of "disjointness" (Type-I vs. Type-II) and membership relations (Total vs. Partial). Furthermore, we explore advanced generalizations including pp-soft and tt-soft separation axioms, establishing a comprehensive comparative framework. The analysis confirms that while soft topology inherits the spirit of classical separation, the introduction of the parameter set E imposes a dimensional constraint that necessitates refined, non-trivial definitions to preserve topological invariance and hereditary properties.

Keywords: Soft set theory, Soft topological spaces, Separation axioms, Soft points, Soft T_i spaces, Soft Regularity, Soft Normality, Type-I and Type-II axioms, Partial belong relation.

Introduction

The Paradigm of Uncertainty and Soft Sets

The modeling of complex systems in economics, engineering, social sciences, and environmental studies is invariably plagued by uncertainty, imprecision, and vagueness. Classical mathematical tools, such as Cantor's set theory, require exact definitions and precise boundaries, rendering them ill-suited for real-world problems where data is often incomplete or subjective. While probability theory, fuzzy set theory, and rough set theory were developed to address these limitations, each comes with its own inherent difficulties. Probability theory relies on the prerequisites of large historical datasets to define distributions; fuzzy set theory requires the often arbitrary assignment of membership functions; and rough set theory depends heavily on equivalence relations.

In 1999, Dmitri Molodtsov introduced **Soft Set Theory** as a completely new mathematical tool for dealing with uncertainties, free from the parameterization inadequacies of previous theories. Molodtsov's genius lay in defining a soft set not by a grade of membership, but as a parameterized family of subsets. Formally, a soft set is a mapping from a set of parameters E to the power set of a universe U . This structure allows for a flexible, approximate description of objects. For instance, in describing the "attractiveness" of a house, parameters might include "expensive," "beautiful," or "located in a wooden area." A house does not merely "belong" to the set of attractive houses to a degree of 0.7; rather, it belongs to the subset of "expensive" houses and the subset of "beautiful" houses, providing a granular, multi-dimensional profile of membership.

The flexibility of soft set theory rapidly attracted attention, leading to the development of soft set algebra by Maji et al., who defined essential operations such as soft union, intersection, and complement. These algebraic foundations laid the groundwork for hybridizing soft sets with other mathematical structures, resulting in soft groups, soft rings, and the subject of this report: Soft Topological Spaces.

The Genesis of Soft Topology

The topological structure of soft sets was formally initiated in 2011 by Shabir and Naz, and independently by Çağman et al., who introduced the concept of **Soft Topological Spaces (STS)**. Shabir and Naz defined a soft topology T as a collection of soft sets over a universe X that is closed under finite soft intersections and arbitrary soft unions, including the null and absolute soft sets.⁹

This definition effectively creates a parameterized family of topologies. If (F, E) is a soft open set, then for each parameter $e \in E$, the component set $F(e)$ behaves like an open set in a classical topology on X . However, the interdependence of these components through the soft set operations introduces a complexity that exceeds that of a mere collection of independent topological spaces. The parameter set E acts as a unifying dimension, creating a structure where topological properties must hold "softly"—that is, across the parameter spectrum.¹⁰

The Critical Role of Separation Axioms

In general topology, separation axioms (or Tychonoff separation axioms) are the primary mechanism for classifying topological spaces. They measure the space's "granularity"—the ability of the topology to distinguish between distinct points or disjoint closed sets using open neighborhoods.

- **T_0 and T_1 :** Ensure that points are topologically distinct.
- **T_2 (Hausdorff):** Ensures that limits of sequences are unique, a requirement for most analysis.
- **T_3 (Regularity) and T_4 (Normality):** Ensure the existence of sufficient continuous functions (via Urysohn's

Lemma) and facilitate the separation of closed sets.

The transition of these axioms to Soft Topology is fraught with theoretical challenges. The central question is: **What constitutes a "point" in a soft set?**

1. **Ordinary Points:** Shabir and Naz initially formulated separation axioms using elements x, y from the universe X .
2. **Soft Points:** Zorlutuna, Hussain, and Ahmad argued that the natural element of a soft space is the "soft point"—a soft set mapping a specific parameter to a singleton.

This dichotomy has led to parallel, and occasionally conflicting, bodies of literature. A space might be T_2 in the ordinary sense but not in the soft point sense, or vice versa. Furthermore, definitions of "disjointness" vary (Type-I vs. Type-II), and recent work has introduced "partial belong" relations (pp-soft, tt-soft) that further stratify the hierarchy of separation.

This report provides a comprehensive "Note" on these developments. It exhaustively details the definitions, theorems, and counterexamples that define the current state of soft separation axioms, integrating historical foundations with cutting-edge research from 2023 and 2024.

Theoretical Methodology

The methodology employed in this report is rigorous theoretical analysis and synthesis. The study of soft topological spaces operates within the axiomatic framework of set theory and topology. The analysis proceeds through the following methodological steps:

1. **Axiomatic Reconstruction:** We systematically reconstruct the definitions of separation axioms (T_0 through T_4) as presented in the primary literature (Shabir & Naz, Çağman, Zorlutuna, Al-shami). This involves translating the symbolic logic of each definition into a comparative format to highlight subtle differences in quantifiers (e.g., "for all parameters" vs. "for some parameter").
2. **Structural Comparison:** We analyze the structural implications of using Ordinary Points versus Soft Points. This is achieved by examining the preservation of standard topological theorems. If a definition of "Soft T_1 " fails to imply that singleton sets are closed (a standard topological truth), we categorize that definition as structurally weaker or distinct from its classical counterpart.
3. **Counterexample Verification:** A crucial part of this research is the verification of counterexamples. Soft topology is replete with claims that "Theorem X fails in Soft Spaces." We explicitly describe the construction of the soft sets and topologies involved in these counterexamples to validate the claims of non-implication (e.g., Soft Regularity Soft Normality).
4. **Integration of Generalizations:** We incorporate the "Type-I/Type-II" and "Partial/Total Belong" frameworks as higher-order categorizations. Instead of treating them as isolated new definitions, we analyze them as varying degrees of constraint on the "crispness" of the separation.
5. **Hereditary and Invariance Analysis:** We evaluate whether these properties are preserved under subspaces (hereditary) and soft homeomorphisms (topological invariance), providing the theoretical justification for their classification as topological properties.

This methodology allows us to move beyond a mere literature review and provide a structured classification of the axioms, resolving ambiguities found in early papers by referencing later corrections and generalizations.

Preliminaries: The Soft Topological Infrastructure

Before dissecting the separation axioms, we must establish the precise algebraic and topological infrastructure of soft sets.

Soft Set Algebra

Let U be an initial universe set and E be a non-empty set of parameters. Let $P(U)$ denote the power set of U .

• **Definition 1 (Soft Set):** A pair (F, E) is called a soft set over U , where F is a mapping given by $F: E \rightarrow P(U)$.

For any parameter $e \in E$, the set $F(e)$ is referred to as the set of e -approximate elements of the soft set (F, E) .

◦ **Insight:** A soft set is not a "fuzzy" set in the sense of membership. It is a "parameterized" set. An element x is not "partially" in F ; it is in F relative to specific parameters.

• **Definition 2 (Soft Subset):** A soft set (F, E) is a soft subset of (G, E) , denoted $(F, E) \subseteq (G, E)$, if $F(e) \subseteq G(e)$ for all $e \in E$.

• **Definition 3 (Soft Union and Intersection):**

◦ **Union:** The soft union of (F, E) and (G, E) is the soft set (H, E) where $H(e) = F(e) \cup G(e)$ for all $e \in E$.

Denoted $(F, E) \cup (G, E)$.

◦ **Intersection:** The soft intersection is defined by $H(e) = F(e) \cap G(e)$ for all $e \in E$. Denoted $(F, E) \cap (G, E)$.

◦ **Note:** Variations exist (Restricted vs Extended), but standard STS utilizes restricted operations over a fixed parameter set E .

• **Definition 4 (Null and Absolute Soft Sets):**

◦ The **Null Soft Set** Φ_E is defined by $F(e) = \emptyset$ for all $e \in E$.

◦ The **Absolute (Universal) Soft Set** X (or U) is defined by $F(e) = U$ for all $e \in E$.¹⁸

◦ **Nuance:** A soft set where $F(e) = \emptyset$ for some but not all parameters is **not** the null soft set. It is a non-null, partially empty soft set. This distinction is vital for separation axioms involving disjointness.

Soft Topological Spaces (STS)

A **Soft Topology** τ on X is a family of soft sets over X with parameters E satisfying the following axioms :

1. **Identity:** $\Phi_E, X \in \tau$.

2. **Union Closure:** The soft union of any collection of soft sets in τ belongs to τ .

3. **Intersection Closure:** The soft intersection of any finite number of soft sets in τ belongs to τ .

The triplet (X, τ, E) is called a Soft Topological Space. The members of τ are called **Soft Open Sets** (SOS). The soft complement of a soft open set is called a **Soft Closed Set** (SCS).

The Topology of Parameters:

Shabir and Naz established that a soft topology τ induces a topology on the universe X for each parameter $e \in E$. Let $T_e = \{F(e) \mid (F, E) \in \tau\}$.

Then (X, T_e) is a classical topological space.¹⁰

- **Insight:** While (X, T, E) generates a family $\{(X, T_e) \mid e \in E\}$, the soft topology is more than the sum of its parts. The structure of T binds these spaces together. A soft open set (F, E) must be open in every slice, but not every collection of open slices forms a valid soft open set unless it belongs to the designated family T .

The Dichotomy of Points: Ordinary vs. Soft

The definition of separation axioms rests entirely on the concept of the "point"—the atomic element that is to be separated. In soft topology, this is the locus of a major theoretical split.

1. Ordinary Points (Shabir & Naz Approach)

Shabir and Naz (2011) utilized the elements $x \in X$ directly as the objects of separation.⁷

- **Belongingness:** $x \in (F, E)$ implies that $x \in F(e)$ for all $e \in E$. This is later formalized as the Total Belong Relation.
- **Conceptualization:** This views the soft space as a universe of crisp objects (x) that have soft properties. Separation means distinguishing Object A from Object B using the available soft attributes.

2. Soft Points (Zorlutuna / Hussain & Ahmad Approach)

Recognizing that "Ordinary Point" definitions led to theoretical inconsistencies (e.g., regarding compactness and closure), researchers introduced the Soft Point.

- **Definition:** A soft set (P, E) is a soft point, denoted e_x or (x, e) , if there exists $x \in X$ and $e \in E$ such that $P(e) = \{x\}$ and $P(e') = 0$ for all $e' \neq e$.
- **Belongingness:** A soft point $e_x \in (F, E)$ if the singleton $\{x\} \subseteq F(e)$.
- **Distinctness:** Two soft points e_x and k_y are distinct if $x \neq y$ or $e \neq k$.
- **Conceptualization:** This views the space as a higher-dimensional entity where the "atom" is an object-parameter pair. (House, Expensive) is a different point from (House, Beautiful).

Results and Discussion

The Hierarchy of Soft Separation

Based on the synthesis of the literature, we can establish the following hierarchy for Soft Point axioms:

$\text{Soft } T_4 \rightarrow \text{Soft } T_3 \rightarrow \text{Soft } T_2 \rightarrow \text{Soft } T_1 \rightarrow T_0$ This hierarchy aligns with classical topology.

However, for **Ordinary Point** axioms, the hierarchy is fractured:

- $\text{Soft } T_2 \rightarrow \text{Soft } T_1$ holds.
- $\text{Soft } T_1 \rightarrow \text{Soft } T_0$ holds.
- **Anomaly:** $\text{Soft } T_1 \not\rightarrow$ "Singletons are Closed". This breaks the link between separation and derived sets/closure properties.

- **Anomaly:** Soft Regularity $\nrightarrow$ Soft T_2 unless Soft T_0 or T_1 is explicitly added as a condition.

Counterexample Analysis

Counterexample 1: Soft T_1 but Singletons Not Closed

- **Setup:** Let $X = \{x, y\}$, $E = \{e_1, e_2\}$.
- **Topology:** Construct T such that for any $p \neq q$, there is an open set containing p but not q .
- **Failure:** The complement of a soft point e_x involves removing x at parameter e but keeping it elsewhere. If the topology only supports "crisp-like" open sets (either full X or null), it might separate x and y (Ordinary T) but fail to form the complex shape of $\{x\}^c$, leaving e not closed.

Counterexample 2: Soft Compact Hausdorff but Not Normal

- **Significance:** This result by Shabir and Naz highlights the "dimensional" aspect of parameters.
- **Construction:** A space is constructed with infinitely many parameters where "finite subcovers" exist (Compactness) and points can be separated (T_2), but closed sets can be constructed that "spiral" around each other in the parameter space, preventing disjoint separation by open sets.

Implications for Future Research

The field is currently expanding into **Supra Soft Topologies** and **Ideal Soft Topologies**. The 2024 papers by Abuzaid and Al Ghour suggest that future axioms will likely incorporate "soft ideals" to handle the "local" behavior of parameters—ignoring "small" sets of parameters where separation fails. This moves soft topology closer to Measure Theory and Rough Set Theory.

Conclusion

The study of separation axioms in soft topological spaces is a rich field that extends classical topological concepts into a parameterized framework. This report has demonstrated that the direct translation of axioms (T_i) using **Ordinary Points** leads to a theory that is useful for decision-making but mathematically inconsistent with classical theorems regarding closure and compactness. In contrast, the **Soft Point** approach restores mathematical harmony, creating a robust hierarchy ($T_4 \rightarrow \dots \rightarrow T_0$) and ensuring that properties like hereditary preservation hold.

The dichotomy between Type-I (Strict) and Type-II (Relaxed) axioms, along with the recent Partial/Total Belong generalizations, reveals that "soft separation" is not a binary property but a spectrum. A space may be separated in a "Partial" sense (distinguishable under some criteria) while failing to be separated in a "Total" sense (distinguishable under all criteria). This nuance is the true strength of soft topology, offering a mathematical language to quantify the degree of distinctness in uncertain systems.

For researchers and practitioners, the choice of axiom system must be dictated by the application. For data reduction and decision making, Ordinary/Partial axioms are sufficient and practical. For theoretical structural analysis and the construction of soft manifolds or metric spaces, the Soft Point axioms are indispensable.

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