Fuzzy Topological Ordered Spaces in Approximation Theory

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Abstract :- In this paper by combining the notions of certain types of fuzzy order and fuzzy topology, we introduce fuzzy topological fuzzy ordered spaces. Its various properties are analyzed. Since the fuzzy topological space may be considered as a fuzzy topological ordered space when it is realized that the non-empty set X is partially ordered by agreeing that $x \leq y$ in X if and only if $x = y$. Then the study of the fuzzy topological ordered spaces not only includes the study of the abstract fuzzy topological spaces but also reveals many generalizations of well-known results concerning the abstract fuzzy topological spaces. We also develop and study order separation axioms called FTi separation axioms for " fuzzy topological ordered spaces in approximation theory." The relationship between some of these FTi separation axioms are also studied.
Keywords :- Fuzzy order, Fuzzy topology, separation axioms, etc.

(1). Introduction :- The concept of a fuzzy set was given by Zadeh. This cocept has been applied by many authors to many branches of mathematics. One of these applications was the study of the fuzzy topological spaces. L. Nachbin in his famous book " Topology and Order" published in 1965 studied the relationship between topological and ordered structures. Several other authors have continued the work of Nachbin on the above relationship. To study of the independence between fuzzy topology and order , Katsaras introduced fuzzy topological ordered spaces in 1981. We think that we will get better result on relationship fuzzy topology and order if the order defined on X in a fuzzy order. So, we define a fuzzy topological fuzzy ordered space. This space is a generalization of Katsars's fuzzy topological ordered spaces as well as fuzzy topological spaces.

(2). Preliminaries :- Let X be a non-empty set. A preorder on X is a relation $\leq$ on X which is reflexive and transitive. A preorder on X which is also anti-symmetric is called a partial order or simply an order. By preordered set we mean a set X with a preorder on it and we denote it by $(X, \leq)$. A preordered set on which there is given a fuzzy topology is called a fuzzy topological preordered space and we denote it by $(X, \leq, \tau)$.

Definition (2.1) :- A fuzzy set μ , in a pre-ordered set $(X, \leq)$, is called

(i) Increasing if $x \leq y$ implies $\mu(x) \leq \mu(y)$;

(ii) Decreasing if $x \leq y$ implies $\mu(x) \geq \mu(y)$;

(iii) Order-convex if $x \leq y \leq z$ implies $\mu(y) \geq \min \{\mu(x), \mu(y)\}$.

The smallest increasing fuzzy set, in $(X, \leq)$, which contains μ is called the increasing hull denoted by $i(\mu)$. The decreasing hull $d(\mu)$ is defined analogously.

Proposition (2.2) :- Let μ be fuzzy set in a preordered set $(X, \leq)$. Then, for each $x \in X$, we have

(i) $i(\mu)(x) = \sup\{ \mu(y) : y \leq x \}$;

(ii) $d(\mu)(x) = \sup\{ \mu(y) : y \geq x \}$;

(iii) $c(\mu)(y) = \sup\{\min \{ \mu(x), \mu(z) \} : x \leq y \leq x \}$.

Definition (2.3) :- Let μ be a fuzzy set, in a preordered set (X, ≤) with a fuzzy topology τ. Then we have

(i) $D(\mu) = \inf \{ \rho: \rho \geq \mu, \rho \text{ is closed and decreasing } \}$;

(ii) $I(\mu) = \inf \{ \rho: \rho \geq \mu, \rho \text{ is closed and increasing } \}$;

Clearly, $D(\mu)$ is the smallest closed decreasing fuzzy set in (X, ≤) which contains μ .

A fuzzy set μ , in a fuzzy topological space (X, τ), is called a neighbourhood of a point x ϵ X if there exists a τ – open fuzzy set ρ with $\rho \leq \mu$ and $\rho(x) = \mu(x) > 0$. It shown that a fuzzy set μ is τ-open if and only if μ is a neighbourhood of each x ϵ X for which $\mu(x) > 0$. A set X with a fuzzy topology τ on it is called a fuzzy topological space. The members of τ in X is called closed if $(1-\mu)$ is open. A fuzzy set μ , in a fuzzy topological space X, is called a neighbourhood of a point x ⊂ X, if there exists an open fuzzy set μ_1 with $\mu_1 \leq \mu$ and $\mu_1(x) = \mu_1 \mu(x) > 0$. It is shown that a fuzzy set μ is open if and only if μ is a neighbourhood of each x ϵ X for which $\mu(x) > 0$.

(3). Fuzzy T_1 Ordered Space :-

Definition (3.1) :- A fuzzy topological ordered space (X, ≤, τ) is said to be an upper fuzzy T_1-ordered space if and only if for each pair of elements x, y ϵ x ⋦ y, there exists a decreasing τ-open neighbourhood μ of y such that $\mu(x) = 0$. (X, ≤, τ) is said to be an fuzzy T_1-ordered space if and only if it is both an upper and a lower fuzzy T_1-ordered space.

Proposition (3.2) :- A fuzzy topological ordered space (X, ≤, τ) is fuzzy T_1-ordered if and only if , for which each x ϵ X, $d(l_x)$ and $i(l_x)$ are τ-closed.

Proof :- First, suppose that (X, ≤, τ) is an upper fuzzy T_1-ordered space. Then for a pair x⋦ y in X, there exists a decreasing τ-open neighbourhood μ of y such that $\mu(x) = 0$. For each y ϵ X\{x}, we have that $[1-i(l_x)](y) > 0$ and $\mu \leq [1-i(l_x)]$ i.e. $[1-i(l_x)]$ is a neighbourhood of each y so that $i(l_x)$ is τ-closed. Similarly $d(l_x)$ is τ-closed.

Conversely :- Let $d(l_x)$ and $i(l_x)$ be τ-closed for each x ϵ X. Then for x ⋦ y, $[1- i(l_x)]$ is a

decreasing τ-open neighbourhood of y where $[1- i(l_x)](x) = 0$, which means that (X, ≤, τ) is upper fuzzy T_1-ordered.

(4). Fuzzy T_2-ordered Spaces :-

Definition (4.1) :- Let (X, ≤, τ) be a fuzzy topological ordered space, then the following conditions are equivalent:

(i) (X, ≤, τ) is fuzzy T_2-ordered.

(ii) For each pair of elements x, y ϵ X with x ⋦ y, there are τ-open neighbourhoods ρ_1 and ρ_2 of x and y, respectively, such that $i(\rho_1) \wedge d(\rho_2) = 0$.

(iii) $X_x \to x$, $y_x \to y$ and $x_x \leq y_x$ implies $x \leq y$.

Proof :- (i) ⟺ (ii) is immediate.

(ii) ⟺ (iii) see next proposition.

Definition (4.2) :- A fuzzy topological ordered space (X, ≤, τ) is said to be fuzzy T_2-ordered space if and only if for each pair x, y ϵ X with x ⋦ y, there is an increasing τ-open neighbourhood μ of x and a decreasing τ-open neighbourhood λ such that $\mu \wedge \lambda = 0$.

Proposition (4.3) :- Every fuzzy T_2-ordered space is a fuzzy T_1-ordered space.

Proof :- Let $Y \subset X$, where $(X, \leq, \tau)$ is a fuzzy topological ordered space. Then the triple $(Y, \leq_y, \tau_y)$ consisting of the subset $Y \subset X$, the induced relative order $\leq$ and the relative fuzzy topology $\tau_y = \{ \delta/Y : \delta \in \tau \}$ is called a fuzzy topological ordered subspsce of $(X, \leq, \tau)$. The fuzzy topological ordered subspaces $(Y, \leq_y, \tau_y)$ is called τ-compatibly ordered if and only if for each τ_y -closed increasing fuzzy set λ , there exists a τ-closed increasing fuzzy set λ^* such that $\lambda = \lambda^*/Y$.

(5). Fuzzy Perfectly Normal Ordered Spaces :-

Definition (5.1) :- A fuzzy topological ordered space $(X, \leq, \tau)$ is said to be fuzzy perfectly normal ordered space if and only if it is fuzzy normal and every decreasing τ-closed fuzzy set is a countable intersection of decreasing τ-open fuzzy sets.

Proposition (5.2) :- A fuzzy topological ordered space $(X, \leq, \tau)$ is called an fuzzy perfectly normal-ordered space if and only if for every decreasing τ-closed fuzzy set μ and a decreasing τ-open fuzzy set ρ such that $\mu \leq \rho$, there exists an order preserving continuous function $f : X \rightarrow [0,1]_\Phi$, such that, for each $x \in X$.

$$\mu(x) = 1\text{-}f(x)(0+) \leq 1 - f(x)(1-) = \rho(x).$$

(6). Fuzzy Regularly Ordered Spaces :-

Definition (6.1) :- A fuzzy topological ordered space $(X, \leq, \tau)$ is said to be upper regularly ordered space if and only if the following condition holds: Given a point $x \in X$ and decreasing τ-open fuzzy neighbourhood λ of x , there exists a decreasing τ-closed fuzzy set ν and a

τ-open decreasing fuzzy set μ such that $\mu(x) > 0$ and $\mu \leq \nu \leq \lambda$.

$(X, \leq, \tau)$ is said to be a regularly ordered space if and only if it is both lower and upper regularly ordered.

Proposition (6.2) :- A fuzzy topological ordered space $(X, \leq, \tau)$ is lower fuzzy regularlyordered if and only if the following condition holds: For each point $x \in X$ and an increasing τ-open fuzzy neighbourhood μ of x, there exists an increasing τ-open fuzzy set λ such that $\lambda(x) > 0$ and $\lambda \leq D(\lambda) \leq \mu$.

Definition (6.3) :- An fuzzy regularly-ordered space which is also fuzzy T_1-ordered is called an fuzzy T_3-ordered space.

(7). Fuzzy Normally Ordered Spaces :-

Definition (7.1) :- A fuzzy topological ordered space $(X, \leq, \tau)$ is called a fuzzy normally orderd space if and only if the following condition is satisfied : Given a decreasing τ-closed fuzzy set μ and a decreasing τ-open fuzzy set ρ such that $\mu \leq \rho$, there are a decreasing τ-open fuzzy set λ and a decreasing τ-closed fuzzy set ν such that $\mu \leq \lambda \leq \nu \leq \rho$.

Theorem (7.2) :- Let $(X, \leq, \tau)$ be a fuzzy topological ordered space, then the following conditions are equivalent:

(i) $(X, \leq, \tau)$ is fuzzy normally-ordered.

(ii) Given a decreasing τ-closed fuzzy set μ and a decreasing τ-open fuzzy set ρ with $\mu \leq \rho$, there exists a decreasing τ-open fuzzy set λ such that $\mu \leq \lambda \leq D(\lambda) \leq \rho$.

(iii) Given a decreasing τ-closed fuzzy set μ and a decreasing τ-open fuzzy set ρ with $\mu \leq \rho$, there exists an order preserving continuous function

$F : X \rightarrow [0,1]_\Phi$, such that, for each $x \in X$ $\mu(x) \leq 1$
$- f(x)(0+) \leq 1 - f(x)(1-) \leq \rho(x)$.

Proposition (7.3) :- An fuzzy normally-ordered space is an fuzzy regularly-ordered space.

Proof :- Let x ϵ X be a point, μ be decreasing τ-closed fuzzy set and ρ be a decreasing τ-open neighbourhood of x with $\mu \leq \rho$. Fuzzy normality of (X, ≤, τ) implies that there exists a decreasing τ-open fuzzy set λ such that $\mu \leq \lambda \leq D(\lambda) \leq \rho$.

So that (X, ≤, τ) is fuzzy regularly-ordered.

(8). Ordered Fuzzy Topological Spaces :- A preordered set on which there is given a fuzzy topology is called a preordered topological space.

Definition (8.1) :- A preordered fuzzy topological space X is called :

(i) Locally order-convex if for each x ϵ X, the collection of all order-convex neighbourhoods of x is a base for the system of all neighbourhoods of x.

(ii) Order-convex if the family of all open order-convex fuzzy sets is a base for the topology.

It is clear that an order-convex fuzzy topological space is locally order convex.

Proposition (8.2) :- Let X be a preordered fuzzy topological space and suppose that the family of all open decreasing and all open increasing fuzzy sets in X is a subbase for the topology. The X is order-convex.

Proof :- If μ_1, ….. μ are open increasing fuzzy sets in X. then $\mu_1 \wedge$ …. $\wedge \mu_n$ is an open increasing fuzzy set. Thus, the hypothesis implies that the family of all fuzzy sets in X of the form $\mu_1 \wedge \mu_2$ with μ_1 open increasing and μ_2 open increasing, is a base for the topology of X.

(9). Conclusion :- In this paper, we conclude that the fuzzy topological ordered spaces contribute to approximation theory by providing a framework to model and analyze uncertain and ordered data using fuzzy sets and topological structures. The conclusions from research highlight their utility in developing fuzzy models for various systems, such as creating fuzzy approximation spaces and classifying fuzzy topological spaces with specific properties . This integration allows for better handling of vagueness in real world scenarios, leading to more robust decision-making processes and applications in fields like sets and beyond.

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