ON ROUGH IDEALS IN Γ-NEAR-RINGS
Dr. V.S. Subha
Assistant Professor, Department of Mathematics
Annamalai University, Annamalainagar-608002, India

Abstract
The aim of this paper is to present the concepts of congruence relation in Γ-near-rings and the lower and upper approximations of an ideal with respect to the congruence relation and introduce the notion of rough ideals in Γ-near-rings, which is a generalization of the notion of near rings. Some properties of the lower and upper approximations are discussed in Γ-near-rings.

Keywords: Γ-near-rings, Congruence relation, Rough ideals.

1 Introduction
Γ-near-ring and the ideal theory of Γ-near-ring were introduced by Bh. Sathyanaranjan[7]. For basic terminology in near-ring we refer to Pilz[6] and in Γ-near-ring.

Pawlak [3-5] introduced the theory of rough sets in 1982. It is another independent method to deal the vagueness and uncertainty. Pawlak used equivalence class in a set as the building blocks for the construction of lower and upper approximations of a set. Many researchers studied the algebraic approach of rough sets in different algebraic structures such as [1, 2, 8, 9]. Thillaigovindan and Subha[10] introduces rough ideals in near-rings.

The aim of this paper is to present the concepts of congruence relation in Γ-near-rings and the lower and upper approximations of an ideal with respect to the congruence relation and introduce the notion of rough ideals in Γ-near-rings, which is a generalization of the notion of near rings. Some properties of the lower and upper approximations are discussed in Γ-near-rings.

2 Preliminaries and Congruence Relation
We first recall some basic concepts for the sake of completeness. Recall from[7], that a non empty set N with two binary operations + and • multiplication is called a near-ring if it satisfies the following axioms.
(i) (N, +) is a group; (ii) (N, •) is a semigroup; (iii) \((n_1 + n_2)n_3 = n_1n_3 + n_2n_3\), for all \(n_1, n_2, n_3 \in N\).

Definition 2.1. [7] A Γ-near-ring is a triple where \((M, +, Γ)\) where

i) \((M, +)\) is a group

ii) \(Γ\) is non empty set of binary operators on M such that for \(α \in Γ\), \((M, +, α)\) is a near-ring

iii) \(xα(xβz) = (xαy)βz\) for all \(x, y, z \in M\) and \(α, β \in Γ\).

In Γ-near-ring, \(0x = 0\) and \((-x)y\) = \(-xy\), but in general \(xy0 \neq 0\) for some \(x \in M, y \in Γ\). More precisely the above near-ring is right near-ring.

\[M_0 = \{n \in M / n0y = 0\}\] is called the zero-symmetric part of M and

\[M = \{n \in M / n0y = n, \text{ for all } y \in Γ\} = \{n \in M / nyn = n \text{ for all } n' \in M, y \in Γ\}\] is called the constant part of M. M is called zero-symmetric if \(M = M_0\) and M is called constant if \(M = M_c\).

Definition 2.2. A subset I of a Γ-near-ring M is called a left(resp. right) ideal of M , if

i) \((I, +)\) is a normal divisor of \((M, +)\) and

ii) \(aa(x + b) = aab \in I\) (resp. \(xyb \in I\)) for all \(x \in I, a \in Γ\) and \(a, b \in M\).

Let I be an ideal of M and X be a non-empty subset of M. Then the sets \(\rho_1(X) = \{x \in M / x + 1 \subseteq X\}\) and \(\overline{\rho}(X) = \{x \in M / (x + 1) \cap X \neq ∅\}\) are called respectively the lower and upper approximations of the set X with respect to the ideal I.

For any ideal I of M and \(a, b \in M\), we say a is congruent to b mod I, written as \(a \equiv b (mod A)\) if \(a - b \in I\).

It is easy to see that relation \(a \equiv b (mod A)\) is an equivalence relation. Therefore, when \(U = M\) and \(θ\) is the above equivalence relation, we use the air \((M, A)\) instead of the approximation space \((U, θ)\).
Also, in this case we use the symbols $\rho_l(X)$ and $\rho_u(X)$ instead of $\rho(X)$ and $\rho(X)$. If $X$ is a subset of $M$, then $X^c$ will be denoted by $M - X$.

3. SOME PROPERTIES OF ROUGH APPROXIMATIONS

In this section we study some fundamental properties of the lower and upper approximations of any subsets of a $\Gamma$-near-ring with respect to an ideal. Throughout this paper $M$ denotes the $\Gamma$-near-ring unless otherwise specified.

**Lemma 3.1.** For every approximation space $(M, I)$ and every subsets $X, Y \subseteq M$, the following hold:

1. $\rho_l(M - X) = M - \rho_u(X)$
2. $\rho_u(M - X) = M - \rho_l(X)$
3. $\rho_u(X) = (\rho_l(X^c))^c$
4. $\rho_l(X) = \left(\rho_u(X^c)\right)^c$

**Proof.** Straight forward.

**Theorem 3.2.** For every approximation space $(M, I)$ and every subsets $X, Y \subseteq M$, then the following hold:

1. $\rho_l(X) \subseteq X \subseteq \rho_u(X)$
2. $\rho_l(\emptyset) = \emptyset \subseteq \rho_u(\emptyset)$
3. $\rho_l(M) \subseteq M \subseteq \rho_u(M)$
4. $\rho_u(X \cup Y) = \rho_u(X) \cup \rho_u(Y)$
5. $\rho_l(X \cap Y) = \rho_l(X) \cap \rho_l(Y)$
6. If $X \subseteq Y$, then $\rho_l(X) \subseteq \rho_u(Y)$ and $\rho_u(X) \subseteq \rho_l(Y)$
7. $\rho_l(X \cap Y) \subseteq \rho_l(X) \cap \rho_l(Y)$
8. $\rho_u(X \cup Y) \supseteq \rho_u(X) \cup \rho_u(Y)$
9. If $I$ is an ideal of $M$ such that $I \subseteq f$, then $\rho_l(A) \supseteq \rho_l(A)$ and $\rho_u(A) \subseteq \rho_u(A)$
10. $\rho_l(\rho_l(X)) = \rho_l(X)$
11. $\rho_u(\rho_u(M)) = \rho_u(M)$
12. $\rho_l(\rho_u(M)) = \rho_u(X)$
13. $\rho_u(\rho_l(M)) = \rho_u(M)$
14. $\rho_l(x + 1) = \rho_u(x + 1)$ for all $x \in M$.
15. $\rho_l(X) \Gamma \rho_l(Y) \subseteq \rho_l(X \Gamma Y)$
16. $\rho_l(X) \Gamma \rho_u(Y) \subseteq \rho_l(X \Gamma Y)$

**Corollary 3.4.** Let $(M, I)$ be any approximation space. Then

(i) For every $A \subseteq M$, $\rho_l(A)$ and $\rho_u(A)$ are definable sets

(ii) For every $x \in M, x + I$ is definable set.

**Theorem 3.5.** Let $I$ be an ideal of $M$ and $A, B$ nonempty subsets of $M$, then $\rho_l(A) \Gamma \rho_u(B) = \rho_l(A \Gamma B)$.

**Proof.** Let $x \in \rho_u(A) \Gamma \rho_l(B)$. Then $x = ayb$ for some $a \in \rho_l(A)$ and $\alpha \in \rho_u(B)$. There exist $y, z \in M, y \in I$ such that $y \in (a + I) \cap A$ and $z \in (b + I) \cap B$. Hence $y \alpha z \in A \Gamma B$ and $y \alpha z \in (a + I) \Gamma (b + I)$. This implies that $y \alpha z \in ayb + I = x + I$ and hence $a \in \rho_u(A \Gamma B)$. Hence $\rho_l(A) \Gamma \rho_l(B) \subseteq \rho_l(A \Gamma B)$. \hfill (1)

On the other hand assume that $x \in \rho_l(A \Gamma B)$. Then there exists $y \in M$ such that $y \in x + I$ and $y \in A \Gamma B$. This implies that $y = a_1y_b$ for some $a_1 \in A$ and $b_1 \in B$. Since $x \in y + I = a_1y_b + I = (a_1 + I) \Gamma (b_1 + I)$, $x$ can be expressed as $x = x_1y_2$ for some
Let $I$ be an ideal of $M$ and $A, B$ nonempty subsets of $\bar{N}$, then
\[ \overline{p_I(A) \Gamma p_I(B)} \subseteq \overline{p_I(A \Gamma B)} \]

**Theorem 3.6.** Let $I$ be an ideal of $M$ and $A, B$ nonempty subsets of $\bar{N}$, then
\[ \overline{p_I(A) \Gamma p_I(B)} \subseteq \overline{p_I(A \Gamma B)} \]

**Proof.** Let $x \in M$. Suppose $x \in \overline{p_I(A) \Gamma p_I(B)}$. Then $x = ayb$ for some $a \in p_I(A)$ and $b \in p_I(B)$. Hence $a + I \subseteq A$ and $b + I \subseteq B$. Now $(a + I)(b + I) \subseteq A \Gamma B$ and $ayb + I \subseteq A \Gamma B$. This implies that $x + I \subseteq A \Gamma B$. Hence $x \in \overline{p_I(A \Gamma B)}$.

Thus
\[ \overline{p_I(A) \Gamma p_I(B)} \subseteq \overline{p_I(A \Gamma B)} \]

**Example 3.7.** Let $N = \{0, a, b, c\}$ and $\Gamma = \{0, a, b\}$. Define addition and multiplication in $M$ as follows:

\[
\begin{array}{c|cccc}
+ & 0 & a & b & c \\
\hline
0 & 0 & a & b & c \\
a & a & 0 & c & b \\
b & b & c & 0 & a \\
c & c & b & a & 0 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
\Gamma & 0 & a & b \\
\hline
0 & 0 & 0 & 0 \\
a & 0 & a & a \\
b & 0 & b & b \\
\end{array}
\]

Then $(M, +, \Gamma)$ is a $\Gamma$-near-ring.

Let $I = \{0, a\}, A = \{b, c\}, B = \{a, b\}$. Then $\overline{p_I(A) \Gamma p_I(B)} = \overline{p_I(A \Gamma B)}$.

**Theorem 3.8.** Let $I$ be an ideal of $M$ and $A, B$ nonempty subsets of $M$, then
\[ \overline{p_I(A) \Gamma p_I(B)} = \overline{p_I(A \Gamma B)} \]

**Proof.** Let $x \in \overline{p_I(A) \Gamma p_I(B)}$. Then $x = a + b$ for some $a \in p_I(A)$ and $b \in p_I(B)$. There exist $y, z \in M$ such that $y \in (a + I) \cap A$ and $z \in (b + I) \cap B$.

Now $y + z \in A + B$ and $y + z \in (a + I) + (b + I) = (a + b) + I = x + I$. This shows that $x + I \cap A + B \neq \emptyset$ and hence $x \in \overline{p_I(A + B)}$.

Thus
\[ \overline{p_I(A) \Gamma p_I(B)} \subseteq \overline{p_I(A + B)} \]

(3)

Conversely, assume that $x \in \overline{p_I(A + B)}$. There exists $y \in M$ such that $y \in x + I$ and $y \in A + B$. This implies $y = a_1 + b_1$ for some $a_1 \in A$ and $b_1 \in B$. Since

$x \in y + I = (a_1 + b_1) + I = (a_1 + I) + (a_1 + I)$, $x$ can be expressed as $x = x_1 + x_2$ for some $x_1 \in a_1 + I$ and $x_2 \in b_1 + I$. This means that $a_1 + I \cap A \neq \emptyset$ and $(x_1 + I) \cap A \neq \emptyset$. Hence $x = a_1 + b_1$ and $(x_1 + I) \cap A \neq \emptyset$. This means that $x_1 \in \overline{p_I(A)}$ and $x_2 \in \overline{p_I(B)}$. Thus $x = x_1 + x_2 \in \overline{p_I(A) \Gamma p_I(B)}$.

Hence
\[ \overline{p_I(A \Gamma B)} \subseteq \overline{p_I(A) \Gamma p_I(B)} \]

(4)

Combining (3) and (4), we obtain
\[ \overline{p_I(A + B)} = \overline{p_I(A) \Gamma p_I(B)} \]

**Theorem 3.9.** Let $I$ be an ideal of $M$ and $A, B$ nonempty subsets of $M$, then
\[ \overline{p_I(A) \Gamma p_I(B)} \subseteq \overline{p_I(A + B)} \]

**Proof.** Let $x \in M$. Suppose $x \in \overline{p_I(A) \Gamma p_I(B)}$. Then $x = a + b$ for some $a \in p_I(A)$ and $b \in p_I(B)$. Hence $a + I \subseteq A$ and $b + I \subseteq B$. Now $(a + I)(b + I) \subseteq A + B$ and $(a + I) + (b + I) \subseteq A + B$. This implies that $x + I \subseteq A + B$. Hence $x \in \overline{p_I(A + B)}$, and thus
\[ \overline{p_I(A) \Gamma p_I(B)} \subseteq \overline{p_I(A + B)} \]
The reverse inclusion of the Theorem 3.9 is not true in general which is shown in the following example.

Example 3.10. Consider the same example as in Example 3.7,
\[ \rho_i(A + B) = \{0, a, b, c\}; \]
\[ \rho_i(A) = \{b, c\}, \rho_i(B) = \emptyset; \rho_i(A) + \rho_i(B) = \{b, c\}. \]
Hence \( \rho_i(A + B) \nsubseteq \rho_i(A) + \rho_i(B) \).

Lemma 3.11. Let \( I, J \) be two ideals of \( M \) and \( A \) a nonempty subset of \( M \), then

(i) \( \rho_i(A) \cap \rho_j(A) \subseteq \rho_{i \cap j}(A) \)

(ii) \( \rho_{i \cap j}(A) \subseteq \rho_i(A) \cap \rho_j(A) \).

Proof. (i) Since \( I \cap J \subseteq I, J \) by Theorem 3.2(9) we have, \( \rho_i(A) \subseteq \rho_{i \cap j}(A) \) and \( \rho_j(A) \subseteq \rho_{i \cap j}(A) \). Hence \( \rho_i(A) \cap \rho_j(A) \subseteq \rho_{i \cap j}(A) \).

(ii) Again \( I \cap J \subseteq I, J \) we have, \( \rho_{i \cap j}(A) \subseteq \rho_i(A) \) and \( \rho_{i \cap j}(A) \subseteq \rho_j(A) \).

Hence \( \rho_{i \cap j}(A) \subseteq \rho_i(A) \cap \rho_j(A) \).

This reverse inclusions of Lemma 3.11 are not true in general which is shown in the following example.

Example 3.12. Let \( M = \{0, a, b, c\} \) and \( \Gamma = \{0, a, b, c\} \) Define addition and \( \Gamma \) in \( M \) as follows:

<table>
<thead>
<tr>
<th>+</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>c</td>
<td>b</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>c</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>b</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ \Gamma \]

<table>
<thead>
<tr>
<th>0</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>0</td>
<td>b</td>
</tr>
</tbody>
</table>

Then \((M, +, \Gamma)\) is a \( \Gamma \)-near-ring.

Let \( I = \{0, a\} \) and \( J = \{0, b\} \) and \( A = \{0, a, c\} \). Then \( I \) and \( J \) are ideals of \( M \). \( \rho_i(A) = \{0, a\} \),
\[ \rho_j(A) = \{a, c\}, \rho_J(A) = M, \rho_I(A) = I \] and \( \rho_{i \cap j}(A) = \{0, a, b\} \), \( \rho_i(A) \cap \rho_j(A) = \{a\}, \rho_{i \cap j}(A) = A. \]
\[ \rho_i(A) \cap \rho_j(A) \nsubseteq \rho_{i \cap j}(A) \] and \( \rho_{i \cap j}(A) \nsubseteq \rho_i(A) \cap \rho_j(A) \).

Theorem 3.13. If \( I \) and \( J \) are two ideals (resp. sub near-rings) of \( M \), then \( \rho_j(J) \) is an ideal (resp. sub near-ring) of \( M \).

Proof. Let \( I \) and \( J \) be ideals of \( M \) and \( i, j \in \rho_j(J) \). Then there exist \( p \in (i + I) \cap J \) and \( q \in (j + I) \cap J \). Since \( J \) is an ideal of \( M \), \( p - q \in J \),
\[ p - q \in (i + I) - (j + I) = i + I + I - j \]
\[ \subseteq i + I - j \]
\[ = i - j + (j + I - j) \]
\[ \subseteq i - j + I. \]

This implies that \((i - j) + J \cap I \neq \emptyset\) and so \( i, j \in \rho_j(J) \).

Assume that \( x \in \rho_j(J) \) and \( a \in M \). Then there exists \( p \in (x + I) \cap J \) such that \( p \in x + I \) and \( p \in J \). Since \( J \) is an ideal of \( M, a + p - a \in J \) and
\[ a + p - a \in a + x + I - a = a + x - a + I - a \]
\[ \subseteq a + x - a + I. \]

This shows that \((a + x - a + I) \cap J \neq \emptyset\) and \( a + x - a \in \rho_j(J) \).

Suppose \( p \in \rho_j(J) \) and \( a \in M \). There exists \( j \in M \) such that \( j \in (p + I) \cap J \) being an ideal of \( M, jya \in J \) and \( jya \in (p + I)ya = pya + I. \) Thus \((pya + I) \cap J \neq \emptyset\) and
\[ p \in \rho_j(J). \] Hence \( \rho_j(J) \cap \emptyset \subseteq \rho_j(J). \) Let \( a, b \in M \) and \( p \in \rho_j(J). \) So there exists \( i \in (p + I) \cap J. \) Since \( J \) is an ideal of \( M, ay(b + i) = ayb \in J \) and
\[
y + (b + l) - a(yb + l - a) \subseteq a(y(b + (p + l)) - ayb + l.
\]
Thus \(((a(y(b + p) - ayb) + l) \cap J = \emptyset \) and hence \(a(y(b + p) - ayb) + I) = \emptyset\). Thus \(\bar{p}_I(J)\) is an ideal of \(M\).

**Theorem 3.14.** If \(I\) and \(J\) are two ideals (resp. sub \(\Gamma\)-near-rings) of \(M\), then \(\bar{p}_I(J)\) is an ideal (resp. sub \(\Gamma\)-near-ring) of \(M\).

Proof. Let \(I\) and \(J\) be two ideals of \(M\). Let \(x, y \in \bar{p}_I(J)\). Then \(x + I, y + I \subseteq J\). Since \(J\) is an ideal of \(M\), \((x + I) = (y + I) \subseteq J\) and so \(x - y + I \subseteq J\). Hence \(x - y \in \bar{p}_I(J)\). Assume that \(x \in \bar{p}_I(J)\) and \(a \in M\). This implies that \(x + I \subseteq J\) and \(J\) being an ideal of \(M\), \(a + (x + l) - a \subseteq J\) and \(a + (x + l) - a \subseteq \bar{p}_I(J)\). Let \(x \in \bar{p}_I(J)\) and \(a \in M\). Then \(x + I \subseteq J\) and \((x + l)ya \subseteq J\).

Hence \(xya + I = (x + I)ya \subseteq J\) and \(xya \in \bar{p}_I(J)\).

Again, let \(p \in \bar{p}_I(J)\) and \(ayb \in M\). Then \(p + I \subseteq J\). Now
\[
(a(y(b + p) - ayb) + I = (a\gamma(b+p) + I - ayb) \subseteq J,
\]
because \(J\) is an ideal of \(M\). Hence \(a\gamma(b+p) - ayb \in \bar{p}_I(J)\). Thus \(\bar{p}_I(J)\) is an ideal of \(M\).

4 ROUGH NEAR-RINGS AND IDEALS

In this section we introduce the notion of rough \(\Gamma\)-near-rings and rough ideals and study some of their properties.

**Definition 4.1.** Let \(I\) be an ideal of \(M\) and \(p_I(A) = (p_I(A), \bar{p}_I(A))\) a rough set in the approximation space \((M, I)\). If \(p_I(A)\) and \(\bar{p}_I(A)\) are ideals (resp. sub \(\Gamma\)-near-rings) of \(M\), then we call \(p_I(A)\) rough ideal (resp. \(\Gamma\)-near-ring).

Note that a rough sub-\(\Gamma\)-near-ring is also called a rough \(\Gamma\)-near-ring. Clearly every rough ideal is a rough \(\Gamma\)-near-ring but the converse need not be true in general.

**Lemma 4.2.**

i) Let \(I, J\) be two ideals of \(M\), then \(p_I(J)\) and \(p_I(J)\) are rough ideals.

ii) Let \(I\) be an ideal and \(J\) a sub near-ring of \(M\), then \(p_I(J)\) is a rough near-ring.

Proof. From Theorem 3.13 and Theorem 3.14, (i) and (ii) are clear.

**Remark 4.3.** If \(I\) is not an ideal and \(J\) is an ideal (resp. sub near-ring) of \(M\), then \(p_I(J)\) is not a rough ideal (resp. rough \(\Gamma\)-near-ring) which is shown in the following example.

**Example 4.4.** Let \(M= \{0, a, b, c, x, y\}\) and \(\Gamma= \{0, a, b, c, x\}\) Define addition and \(\Gamma\) in \(M\) as follows.

\[
\begin{array}{cccccc}
+ & 0 & a & b & c & x & y \\
\hline
0 & 0 & a & b & c & x & y \\
a & a & 0 & y & x & c & b \\
b & b & x & 0 & y & a & c \\
c & c & x & y & 0 & b & a \\
x & x & b & c & a & y & 0 \\
y & y & c & a & b & 0 & x \\
\end{array}
\]

\[
\begin{array}{cccccc}
\Gamma & 0 & a & b & c & x \\
\hline
0 & 0 & 0 & 0 & 0 & 0 \\
a & a & a & a & a & a \\
b & b & a & b & c & b \\
c & c & a & c & b & c \\
x & x & 0 & y & x & x \\
\end{array}
\]

Then \((M, +, \Gamma)\) is a \(\Gamma\)-near-ring.

Let \(I = \{a, c\}, J = \{0, x, y\}\). Clearly, \(J\) is an ideal and \(I\) is not an ideal(sub \(\Gamma\)-near-ring). Since \(0 + I = \{a, c\}, a + I = \{0, x\}, b + I = \{x, y\}, c + I = \{0, x\}, x + I = \{b, c\}\) and \(y + I = \{a, c\}\), \(p_I(J) = \{a, b, c\} = \bar{p}_I(J)\). Thus both \(p_I(J)\) and \(\bar{p}_I(J)\) are not ideals (sub near-rings) of \(M\). Hence \(p_I(J)\) is not a rough ideal (resp. rough \(\Gamma\)-near-ring).

**Theorem 4.5.** Let \(I, J\) be two ideals of \(M\) and \(K\) be a sub \(\Gamma\)-near-ring of \(M\). Then
(i) \( \overline{p}_i(K) \cap \overline{p}_j(K) \subseteq \overline{p}_{i+j}(K) \)

(ii) \( \overline{p}_i(K) \cap \overline{p}_j(K) = \overline{p}_{i+j}(K) \).

**Proof.** (i) Let \( x \in \overline{p}_i(K) \cap \overline{p}_j(K) \). Then \( x = p\gamma q \) for some \( p \in \overline{p}_i(K) \) and \( q \in \overline{p}_j(K) \). This means that there exist \( y \in (p + I) \cap K \) and \( z \in (q + J) \cap K \) and so \( y\gamma z \in K \) and \( xy \in (p + I) \cdot (q + J) \). This implies \( y\gamma z \in (p\gamma q) + I + J \). Thus \( (p\gamma q + I + J) \cap K \neq \emptyset \), and \( x \in \overline{p}_{i+j}(K) \). Hence \( \overline{p}_i(K) \cap \overline{p}_j(K) \subseteq \overline{p}_{i+j}(K) \).

(ii) Let \( p\gamma q \in \overline{p}_i(K) \cap \overline{p}_j(K) \) then \( p \in \overline{p}_i(K) \) and \( q \in \overline{p}_j(K) \) and so \( (p + I) \subseteq K \) and \( (q + J) \subseteq K \). Now \( (p + I) \cap (q + J) \subseteq K \) and \( (p\gamma q + I + J) \subseteq K \). This implies \( p\gamma q \in \overline{p}_{i+j}(K) \).

On the other hand, since \( I \subseteq I + J, J \subseteq I + J \), we have by Theorem 3.2(9), \( \overline{p}_{i+j}(K) \subseteq \overline{p}_i(K) \) and

\[
\overline{p}_{i+j}(K) \subseteq \overline{p}_i(K) \cap \overline{p}_j(K) \cap \overline{p}_{i+j}(K) \subseteq \overline{p}_i(K) \cap \overline{p}_j(K).
\]

This means that \( \overline{p}_{i+j}(K) \subseteq \overline{p}_i(K) \cap \overline{p}_j(K) \).

**Theorem 4.4.** Let \( I, J \) be two ideals of \( M \) and \( K \) a sub near-ring of \( M \). Then

(i) \( \overline{p}_{i+j}(K) = \overline{p}_i(K) + \overline{p}_j(K) \)

(ii) \( \overline{p}_{i+j}(K) = \overline{p}_i(K) + \overline{p}_j(K) \).

Proof. (i) Since \( I \subseteq I + J \) and \( J \subseteq I + J \), by Theorem 3.2(9) we have

\( \overline{p}_{i+j}(K) \subseteq \overline{p}_i(K) \) and \( \overline{p}_{i+j}(K) \subseteq \overline{p}_j(K) \).

Thus \( \overline{p}_{i+j}(K) \subseteq \overline{p}_i(K) + \overline{p}_j(K) \).

Conversely assume that \( k \in \overline{p}_i(K) + \overline{p}_j(K) \). Then \( k = x + y \) for some \( x \in \overline{p}_i(K) \) and \( y \in \overline{p}_j(K) \).

This means that \( x + I \subseteq K \) and \( y + J \subseteq K \). Consider

\[
k + I + J = x + y + I + J = x + y + I + J = x + I + y + J \subseteq K + K \subseteq K.
\]

Thus \( k \in \overline{p}_{i+j}(K) \) and so \( \overline{p}_i(K) + \overline{p}_j(K) \subseteq \overline{p}_{i+j}(K) \).

Thus \( \overline{p}_i(K) + \overline{p}_j(K) = \overline{p}_{i+j}(K) \).

(ii) Since \( I \subseteq I + J \) and \( J \subseteq I + J \), by Theorem 3.2(9), we have

\( \overline{p}_i(K) \subseteq \overline{p}_{i+j}(K) \) and \( \overline{p}_j(K) \subseteq \overline{p}_{i+j}(K) \).

Therefore \( \overline{p}_i(K) + \overline{p}_j(K) \subseteq \overline{p}_{i+j}(K) \).

Conversely assume that \( y \in \overline{p}_{i+j}(K) \). Then \( (y + (I + J)) \cap K \neq \emptyset \). Now there exists \( j \in J \) such that

\[
(y + (I + J)) \cap K = (y + j - j + I + J) \cap K \subseteq (y + J) \cap K \subseteq K.
\]

This means that \( y + J \subseteq \overline{p}_i(K) \). Since \( -j \in J \) and \( (J - j) \cap K = J \cap K \neq \emptyset \), being \( 0 \in J \cap K \), we have \( -j \in \overline{p}_j(K) \).

Consider \( y = y + j - j \in \overline{p}_i(K) + \overline{p}_j(K) \). We have \( \overline{p}_{i+j}(K) \subseteq \overline{p}_i(K) + \overline{p}_j(K) \).

Thus \( \overline{p}_i(K) + \overline{p}_j(K) = \overline{p}_{i+j}(K) \).

5.**CONCLUSION**

The theory of \( \Gamma \)-near ring and theory of rough sets have many application in various fields. In this paper is to present the concepts of congruence relation in \( \Gamma \)-near-rings and the lower and upper approximations of an ideal with respect to the congruence relation and introduce the notion of rough ideals in \( \Gamma \)-near-rings, which is a generalization of the notion of near rings. Some properties of the lower and upper approximations are discussed in \( \Gamma \)-near-rings. The definitions and results are extended to rings.
REFERENCES