Undirected Binary Fuzzy Graphs on Composition, Tensor and Normal Products

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Abstract

Undirected binary fuzzy graphs can be obtained from two given undirected binary fuzzy graphs using the operations, cartesian product, composition, tensor and normal products. In this broadsheet, we find the degree of a pinnacle in undirected binary fuzzy graphs formed by these operations in terms of the degree of vertices in the given undirected binary fuzzy diagrams in some particular cases.

Keywords - Cartesian product, Composition, Degree of a vertex, Tensor product, Normal product.

INTRODUCTION

Fuzzy graphs introduced by Rosenfeld in 1975[1-2, 9, 10]. The operations of union, join, cartesian product and composition on two fuzzy graphs were defined by Mordeson. J. N. and Peng. C.S [3-8]. In this paper, we study about the degree of a vertex in undirected binary fuzzy graphs which are obtained from two given undirected binary fuzzy graphs using the operations cartesian product and composition of two undirected binary fuzzy graphs, tensor and normal product of undirected binary fuzzy graphs. In general, the degree of vertices in cartesian product and composition of two undirected binary fuzzy graphs, tensor and normal product of two undirected binary fuzzy graphs $BG_1$ and $BG_2$ cannot be expressed in terms of those in $BG_1$ and $BG_2$. In this paper, we find the degree of vertices in cartesian product, composition, tensor and normal product of $BG_1$ and $BG_2$ in some particular cases.

PRELIMINARIES

Definition 2.1:
A fuzzy subset $\mu$ on a set $X$ is a map $\mu : X \rightarrow [0, 1]$. A map $\beta : X \times X \rightarrow [0, 1]$ is fuzzy relation on $X$ if $\beta(\alpha, \beta) \leq \mu(x) \land \mu(y)$ for all $\alpha, \beta \in X$. $\beta$ is a symmetric fuzzy relation if $\beta(\alpha, \beta) = \beta(\beta, \alpha)$ for all $\alpha, \beta \in X$.

Definition 2.2:
Let $X$ be a non-empty regular set. A binary fuzzy set $B$ in $X$ is an objective having the practice $B= \{ (\alpha, \mu^B(\alpha), \mu^B(\alpha))/\alpha \in X \}$ where $\mu^B: X \rightarrow [0, 1]$ and $\mu^B: X \rightarrow [0, 1]$ are mappings.

Definition 2.3:
A binary fuzzy graph of $BG^* = (V, E)$ is a pair $BG (A, B)$ where $A= (\mu^A_A, \mu^A_B)$ is a binary fuzzy set in $V$ and $B=(\mu^B_B, \mu^B_B)$ is a binary fuzzy set in $V 	imes V$ such that $(\mu^B_B(\alpha, \beta) \leq \mu^A_A(\alpha) \land \mu^A_B(\beta)$ for all $\alpha, \beta \in VXV$, $(\mu^B_B(\alpha, \beta)) \geq \mu^A_B(\alpha))V \mu^A_B(\beta)$ for all $\alpha, \beta \in VXV$, and $(\mu^B_B(\alpha, \beta)) = (\mu^B_B(\alpha, \beta)) = 0$ for all $\alpha, \beta \in VXV-E$.

Figure 1: Undirected Binary Graph
Definition 2.4: Let $A_1 = (\mu_{A_1}^r, \mu_{A_1}^s)$ and $A_2 = (\mu_{A_2}^r, \mu_{A_2}^s)$ be binary fuzzy graph subsets of $E_1$ and $E_2$ respectively. Then the cartesian product of two $BG_1$ and $BG_2$ of graph $BG_1$ and $BG_2$ by $BG_1 \times BG_2 = (A_1 \times A_2, B_1 \times B_2)$ and defined as follows

1. $(\mu_{A_1}^r \times \mu_{A_2}^r)(\alpha_1, \alpha_2) = (\mu_{A_1}^r(\alpha_1) \mu_{A_2}^r(\alpha_2))$

2. $(\mu_{A_1}^s \times \mu_{A_2}^s)(\alpha_1, \alpha_2) = (\mu_{A_1}^s(\alpha_1) \mu_{A_2}^s(\alpha_2))$

Definition 2.5: Let $A_1 = (\mu_{A_1}^r, \mu_{A_1}^s)$ and $A_2 = (\mu_{A_2}^r, \mu_{A_2}^s)$ be an undirected binary fuzzy subgraph of $V_1$ and $V_2$ let $B_1 = (\mu_{B_1}^r, \mu_{B_1}^s)$ and $B_2 = (\mu_{B_2}^r, \mu_{B_2}^s)$ be binary fuzzy graph subsets of $E_1$ and $E_2$ respectively. Then the composition of two binary fuzzy graphs $BG_1$ and $BG_2$ of graphs $BG_1$ and $BG_2$ by $BG_1 \circ BG_2 = (A_1 \circ A_2, B_1 \circ B_2)$ and defined as follows

1. $(\mu_{A_1}^r \circ \mu_{A_2}^r)(\alpha_1, \alpha_2) = (\mu_{A_1}^r(\alpha_1) \mu_{A_2}^r(\alpha_2))$

2. $(\mu_{A_1}^s \circ \mu_{A_2}^s)(\alpha_1, \alpha_2) = (\mu_{A_1}^s(\alpha_1) \mu_{A_2}^s(\alpha_2))$

Definition 2.6: Let $A_1 = (\mu_{A_1}^r, \mu_{A_1}^s)$ and $A_2 = (\mu_{A_2}^r, \mu_{A_2}^s)$ be an undirected binary fuzzy subgraph of $V_1$ and $V_2$ let $B_1 = (\mu_{B_1}^r, \mu_{B_1}^s)$ and $B_2 = (\mu_{B_2}^r, \mu_{B_2}^s)$ be binary fuzzy graph subsets of $E_1$ and $E_2$ respectively. Then the normal product of two binary fuzzy graphs $BG_1$ and $BG_2$ of graphs $BG_1$ and $BG_2$ by $BG_1 \star BG_2 = (A_1 \star A_2, B_1 \star B_2)$ and defined as follows

1. $(\mu_{A_1}^r \star \mu_{A_2}^r)(\alpha_1, \alpha_2) = (\mu_{A_1}^r(\alpha_1) \mu_{A_2}^r(\alpha_2))$

2. $(\mu_{A_1}^s \star \mu_{A_2}^s)(\alpha_1, \alpha_2) = (\mu_{A_1}^s(\alpha_1) \mu_{A_2}^s(\alpha_2))$

Definition 2.7: Let $A_1 = (\mu_{A_1}^r, \mu_{A_1}^s)$ and $A_2 = (\mu_{A_2}^r, \mu_{A_2}^s)$ be an undirected binary fuzzy subgraph of $V_1$ and $V_2$ let $B_1 = (\mu_{B_1}^r, \mu_{B_1}^s)$ and $B_2 = (\mu_{B_2}^r, \mu_{B_2}^s)$ be binary fuzzy graph subsets of $E_1$ and $E_2$ respectively. Then the tensor product of two binary fuzzy graphs $BG_1$ and $BG_2$ of graphs $BG_1$ and $BG_2$ by $BG_1 \otimes BG_2 = (A_1 \otimes A_2, B_1 \otimes B_2)$ and defined as follows

1. $(\mu_{A_1}^r \otimes \mu_{A_2}^r)(\alpha_1, \alpha_2) = (\mu_{A_1}^r(\alpha_1) \mu_{A_2}^r(\alpha_2))$

2. $(\mu_{A_1}^s \otimes \mu_{A_2}^s)(\alpha_1, \alpha_2) = (\mu_{A_1}^s(\alpha_1) \mu_{A_2}^s(\alpha_2))$
Degree of a vertex in the cartesian product

In above definition, for any vertex \((\alpha_1, \beta_1) \in V\)

\[
d_{BG_1 \times BG_2}(\alpha_1, \alpha_2) = \sum_{(\alpha_1, \alpha_2)(\beta_1, \beta_2) \in E} \left[ (\mu_{A_1}^p \wedge \mu_{B_1}^p)(\alpha_1, \alpha_2)(\beta_1, \beta_2), (\mu_{A_2}^n \wedge \mu_{B_2}^n)(\alpha_1, \alpha_2)(\beta_1, \beta_2) \right]
\]

\[
= \sum_{(\alpha_1=\beta_1)(\alpha_2, \beta_2) \in E} \left[ (\mu_{A_1}^p(\alpha_1, \alpha_2) \wedge \mu_{B_1}^p(\alpha_1, \alpha_2))(\beta_1, \beta_2), (\mu_{A_1}(\alpha_1) \vee \mu_{B_1}(\alpha_1))(\beta_1, \beta_2) \right] + \sum_{(\alpha_2=\beta_2)(\alpha_1, \beta_1) \in E} \left[ (\mu_{A_2}^p(\alpha_1, \alpha_2) \wedge \mu_{B_2}^p(\alpha_1, \alpha_2))(\beta_1, \beta_2), (\mu_{A_2}(\alpha_1) \vee \mu_{B_2}(\alpha_1))(\beta_1, \beta_2) \right]
\]

In the following theorems, we define the degree of \((\alpha_1, \alpha_2)\) in \(BG_1 \times BG_2\) in terms of some in particular cases.

**Theorem 1:** Let \(BG_1\) and \(BG_2\) be two undirected binary fuzzy graphs. If \(\mu_{A_1}^p \geq \mu_{B_1}^p, \mu_{A_1}^n \leq \mu_{B_1}^n\) and \(\mu_{A_2}^p \geq \mu_{B_2}^p, \mu_{A_2}^n \leq \mu_{B_2}^n\) then \(d_{BG_1 \times BG_2}(\alpha_1, \alpha_2) = d_{BG_1} + d_{BG_2}\)

**Proof:** By definition of degree of a vertex in cartesian product

\[
d_{BG_1 \times BG_2}(\alpha_1, \alpha_2) = \sum_{(\alpha_1=\beta_1)(\alpha_2, \beta_2) \in E} \left[ (\mu_{A_1}^p(\alpha_1, \alpha_2) \wedge \mu_{B_1}^p(\alpha_1, \alpha_2))(\beta_1, \beta_2), (\mu_{A_1}(\alpha_1) \vee \mu_{B_1}(\alpha_1))(\beta_1, \beta_2) \right] + \sum_{(\alpha_2=\beta_2)(\alpha_1, \beta_1) \in E} \left[ (\mu_{A_2}^p(\alpha_1, \alpha_2) \wedge \mu_{B_2}^p(\alpha_1, \alpha_2))(\beta_1, \beta_2), (\mu_{A_2}(\alpha_1) \vee \mu_{B_2}(\alpha_1))(\beta_1, \beta_2) \right]
\]

\[
= \sum_{(\alpha_1=\beta_1)(\alpha_2, \beta_2) \in E} \left[ (\mu_{A_1}(\alpha_1, \alpha_2) \wedge \mu_{B_1}(\alpha_1, \alpha_2))(\beta_1, \beta_2), (\mu_{A_1}(\alpha_1) \vee \mu_{B_1}(\alpha_1))(\beta_1, \beta_2) \right] + \sum_{(\alpha_2=\beta_2)(\alpha_1, \beta_1) \in E} \left[ (\mu_{A_2}(\alpha_1, \alpha_2) \wedge \mu_{B_2}(\alpha_1, \alpha_2))(\beta_1, \beta_2), (\mu_{A_2}(\alpha_1) \vee \mu_{B_2}(\alpha_1))(\beta_1, \beta_2) \right]
\]

**Example 1**

![Example 1](image)

**Figure 2:** Cartesian product

Here \(\mu_{A_1}^p \geq \mu_{B_1}^p, \mu_{A_1}^n \leq \mu_{B_1}^n\) and \(\mu_{A_2}^p \geq \mu_{B_2}^p, \mu_{A_2}^n \leq \mu_{B_2}^n\) by theorem of 2.

\[
d_{BG_1 \times BG_2}(\alpha_1, \alpha_2) = d_{BG_1} + d_{BG_2}
\]

\[
= (0.6+0.4, 0.3+0.5, 0.6+0.4, 0.3+0.5)
\]

\[
= (1, 0.8, 1, 0.8)
\]

Similarly we find to all vertex in \(BG_1 \times BG_2\) in figure 2

Degree of a vertex in composition product

In above definition, for any vertex \((\alpha_1, \beta_1) \in V\)

\[
d_{BG_1 \times BG_2}(\alpha_1, \alpha_2) = \sum_{(\alpha_1, \alpha_2)(\beta_1, \beta_2) \in E} \left[ (\mu_{A_1}^p \wedge \mu_{A_2}^p)(\alpha_1, \alpha_2)(\beta_1, \beta_2), (\mu_{A_1}^n \wedge \mu_{A_2}^n)(\alpha_1, \alpha_2)(\beta_1, \beta_2) \right]
\]

\[
= \sum_{(\alpha_1=\beta_1)(\alpha_2, \beta_2) \in E} \left[ (\mu_{A_1}^p(\alpha_1, \alpha_2) \wedge \mu_{A_2}^p(\alpha_1, \alpha_2))(\beta_1, \beta_2), (\mu_{A_1}(\alpha_1) \vee \mu_{A_2}(\alpha_1))(\beta_1, \beta_2) \right]
\]

\[
+ \sum_{(\alpha_2=\beta_2)(\alpha_1, \beta_1) \in E} \left[ (\mu_{A_1}(\alpha_1, \alpha_2) \wedge \mu_{A_2}(\alpha_1, \alpha_2))(\beta_1, \beta_2), (\mu_{A_1}(\alpha_1) \vee \mu_{A_2}(\alpha_1))(\beta_1, \beta_2) \right]
\]

**Theorem 2:** Let \(BG_1\) and \(BG_2\) be two undirected binary fuzzy graphs. If \(\mu_{A_1}^p \geq \mu_{B_1}^p, \mu_{A_1}^n \leq \mu_{B_1}^n\) and \(\mu_{A_2}^p \geq \mu_{B_2}^p, \mu_{A_2}^n \leq \mu_{B_2}^n\) then \(d_{BG_1 \times BG_2}(\alpha_1, \alpha_2) = d_{BG_1}(\alpha_1)V_2 + d_{BG_2}(\beta_2)\)

**Proof:** By definition of degree of a vertex in composition product

\[
d_{BG_1 \times BG_2}(\alpha_1, \alpha_2) = \sum_{(\alpha_1, \alpha_2)(\beta_1, \beta_2) \in E} \left[ (\mu_{A_1}^p \wedge \mu_{A_2}^p)(\alpha_1, \alpha_2)(\beta_1, \beta_2), (\mu_{A_1}^n \wedge \mu_{A_2}^n)(\alpha_1, \alpha_2)(\beta_1, \beta_2) \right]
\]

\[
= \sum_{(\alpha_1=\beta_1)(\alpha_2, \beta_2) \in E} \left[ (\mu_{A_1}^p(\alpha_1, \alpha_2) \wedge \mu_{A_2}^p(\alpha_1, \alpha_2))(\beta_1, \beta_2), (\mu_{A_1}(\alpha_1) \vee \mu_{A_2}(\alpha_1))(\beta_1, \beta_2) \right]
\]

\[
+ \sum_{(\alpha_2=\beta_2)(\alpha_1, \beta_1) \in E} \left[ (\mu_{A_1}(\alpha_1, \alpha_2) \wedge \mu_{A_2}(\alpha_1, \alpha_2))(\beta_1, \beta_2), (\mu_{A_1}(\alpha_1) \vee \mu_{A_2}(\alpha_1))(\beta_1, \beta_2) \right]
\]
Example 2:

Here \( d_{BG_1 \otimes BG_2}(\alpha_1, \alpha_2) = d_{BG_2}(\alpha_2) + |V_2|d_{BG_1}(\alpha_1) \)

= 0.4 + 2 (0.3) = 1.0

Similarly we find to all vertex of \( d_{BG_1 \otimes BG_2}(\alpha_1, \alpha_2) \).

Degree of a vertex in tensor product

In above definition, for any vertex \((\alpha_1, \beta_1) \in V_1 \times V_2\)

\[
d_{BG_1 \otimes BG_2}(\alpha_1, \alpha_2) = \sum_{(\alpha_1, \alpha_2) \in V_1 \times V_2} \left[ (\mu^1_{B_1}(\alpha_1, \alpha_2) \wedge \mu^2_{B_2}(\beta_1, \beta_2), (\mu^1_{B_1} \wedge \mu^2_{B_2})(\alpha_1, \alpha_2)(\beta_1, \beta_2) ) \right]
\]

\[= \sum_{(\alpha_1 = \beta_1)(\alpha_2, \beta_2) \in E_2} \left[ (\mu^1_{B_1}(\alpha_1, \alpha_2) \wedge \mu^2_{B_2}(\beta_1, \beta_2), (\mu^1_{B_1} \wedge \mu^2_{B_2})(\alpha_1, \alpha_2)(\beta_1, \beta_2) ) \right]
\]

Theorem 3: Let \( BG_1 \) and \( BG_2 \) be two undirected binary fuzzy graphs. If \( \mu^1_{B_2} \geq \mu^1_{B_1} \) \( \mu^2_{B_1} \leq \mu^2_{B_2} \) then \( d_{BG_1 \otimes BG_2}(\alpha_1, \alpha_2) = d_{BG_1}(\alpha_1) \) and \( \mu^1_{B_2} \geq \mu^1_{B_1} \) \( \mu^2_{B_1} \leq \mu^2_{B_2} \) then \( d_{BG_1 \otimes BG_2}(\alpha_1, \alpha_2) = d_{BG_2}(\beta_2) \)

Proof: By definition of degree of a vertex in cartesian product

\[
d_{BG_1 \otimes BG_2}(\alpha_1, \alpha_2) = \sum_{(\alpha_1, \beta_1) \in E_1} \left[ (\mu^1_{B_1}(\alpha_1, \beta_1) \wedge \mu^2_{B_2}(\alpha_2, \beta_2), (\mu^1_{B_1}(\alpha_1, \beta_1) \wedge \mu^2_{B_2}(\alpha_2, \beta_2) ) \right]
\]

\[= \sum_{(\alpha_1, \beta_1) \in E_1} \left[ (\mu^1_{B_1}(\alpha_1, \beta_1), (\mu^2_{B_2}(\alpha_2, \beta_2) ) \right]
\]

Example 3:

Here \( d_{BG_1 \otimes BG_2}(\alpha_1, \alpha_2) = (0.3, 0.6, 0.6, 0.4) = d_{BG_1}(\alpha_1) \)

\( d_{BG_1 \otimes BG_2}(\alpha_1, \alpha_2) = (0.3, 0.6, 0.6, 0.4) = d_{BG_2}(\beta_1) \)

Degree of a vertex in normal product

In above definition, for any vertex \((\alpha_1, \beta_1) \in V_1 \times V_2\)

\[
d_{BG_1 \times BG_2}(\alpha_1, \alpha_2) = \sum_{(\alpha_1, \alpha_2) \in V_1 \times V_2} \left[ (\mu^1_{B_1}(\alpha_1, \alpha_2) \wedge \mu^2_{B_2}(\beta_1, \beta_2), (\mu^1_{B_1}(\alpha_1, \alpha_2) \wedge \mu^2_{B_2}(\beta_1, \beta_2) ) \right]
\]

\[= \sum_{(\alpha_1 = \beta_1)(\alpha_2, \beta_2) \in E_2} \left[ (\mu^1_{B_1}(\alpha_1, \alpha_2) \wedge \mu^2_{B_2}(\beta_1, \beta_2), (\mu^1_{B_1}(\alpha_1, \alpha_2) \wedge \mu^2_{B_2}(\beta_1, \beta_2) ) \right]
\]
composition, tensor product of degree of vertices in and under some conditions and illustrated them through examples. This will be helpful

THEOREM 4: Let \( BG_1 \) and \( BG_2 \) be two undirected binary fuzzy graphs. If \( \mu_{B_2}^\rho \geq \mu_{B_1}^\rho, \mu_{B_2}^\nu \leq \mu_{B_1}^\nu \) and \( \mu_{A_2}^\rho \geq \mu_{A_1}^\rho \), then \( d_{BG_1 \times BG_2}(\alpha_1, \alpha_2) = d_{BG_1}(\beta_1) + d_{BG_2}(\beta_2) \).

PROOF: By definition of degree of a vertex in cartesian product

\[
d_{BG_1 \times BG_2}(\alpha_1, \alpha_2) = \sum_{(\alpha_1, \alpha_2) \in E} \left[ \mu_{B_1}(\alpha_1, \beta_1) \mu_{B_2}^{\nu}(\beta_2) \right] + \sum_{(\alpha_1, \alpha_2) \in E} \left[ \mu_{A_1}(\alpha_1, \beta_1) \mu_{A_2}^{\nu}(\beta_2) \right] = d_{BG_1}(\alpha_1) + d_{BG_2}(\alpha_2) \]

EXAMPLE 4:

\[
\begin{array}{c}
\alpha_1 \quad \alpha_2 \\
(0.1, 0.4) \quad (0.6, 0.4) \\
\beta_1 \quad (0.3, 0.6) \\
\end{array}
\]

\[
\begin{array}{c}
\alpha_1 \quad \alpha_2 \\
(0.5, 0.4) \quad (0.6, 0.4) \\
\end{array}
\]

Figure 5: Normal product

\[ d_{BG_1 \times BG_2}(\alpha_1, \alpha_2) = (0.3 + 0.3 + 0.6 + 0.6 + 0.4 + 0.4 + 0.4 + 0.4 + 1.8 + 1.8 + 1.8) = (0.9, 1.6, 1.2, 4.2) \]

\[ d_{BG_1 \times BG_2}(\alpha_1, \alpha_2) = 0.3, 0.4, 0.4, 0.4, 0.4, 0.4, 0.4, 0.4, 1.8, 1.8, 1.8) = (0.9, 1.6, 1.2, 4.2) \]

CONCLUSION

In this paper, we have found the degrees of vertices in \( BG_1 \times BG_2, BG_1 \times BG_2, BG_1 \circ BG_2, BG_1 \otimes BG_2 \) in terms of degree of vertices in and under some conditions and illustrated them through examples. This will be helpful when the graphs are very large. Also they will be very useful in studying various properties of cartesian product, composition, tensor product, normal product of two bipolar fuzzy graphs.
REFERENCES