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Comparative Study Of Composition Operator And Sum Of Two Composition Operators On The Banach Space

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Abstract: In this paper, we compare the injectivity and surjectivity of the composition operator and sum of the composition operator, as well as the range and null space of the composition operator and sum of two composition operators on Banach space.

Keywords: Composition operator, Range and Null space of Composition operator,

Injectivity, Surjectivity

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1. INTRODUCTION:

The study of Composition operators leads to an interesting interplay between classical function theory, measure theory, operator theory, and the theory of analytic functions. Composition operators form a simple but diverse and interesting class of operators having interactions with different branches of mathematics and mathematical physics. For the first time, Composition operators appeared in B. O. Koopman's formulations of classical mechanics in 1931 and it was then referred to as Koopman operator in physics, especially in the area of dynamical systems. These operators also found expression, at least implicitly, in connection with Markov processes. These operators were studied in the space of analytic functions by J. V. Ryff in 1966. However, the word "Composition Operator" appeared for the first time in work of Eric Nordgren in

1968.The study of Composition operator was initiated with two goals, first to have concrete examples of bounded operators on Hilbert space or Banach space of functions and second to attack the "Invarient subspace problem" of operator theory from a different angle. Since many concrete examples of the Composition operators on L_p-spaces, H_p-spaces and locally convex spaces of continuous functions have been obtained. It is interesting to note that in certain situations this class of operators possesses a distinct identity different from several known class of operators, like as multiplication operator and the integral operator, differential operators.

2. Composition Operator:

Let X be a non-empty set and V(X) be a linear space of complex valued functions on X under point wise addition and scalar multiplication. If φ is a selfmap on X into itself such that composition $f \circ \varphi \in V(X)$ for each $f \in V(X)$ then φ induces a linear transformation $C\varphi$ on V(X), defined as

$$C\varphi(f) = f \circ \boldsymbol{\varphi} \forall f \in V(X).$$

The transformation $C\varphi$ is known as composition transformation. If V(X) is a Banachor Hilbert space and $C\varphi$is a bounded linear operator on V(X), then $C\varphi$ is called composition operator.

3. Example: Define a self-map φ on $\mathbb{N}$ by

$$\varphi(n) = \begin{cases} n+2, & if\ n\ is\ odd \\ n-2, & if\ n\ is\ even \end{cases}$$

4. Notation and Terminology: For $1 \leq p < \infty$, ℓ_p denotes the space of all

p-summable real or complex sequences and $|\varphi^{-1}(n)|$ denotes the cardinality of function. Further, let $\mathbb{N}$ and $\mathbb{C}$ denote the set of all positive integersand the set of all complex numbers, respectively. For given a subset A of a set X, the characteristic function of A is denoted by $\chi_A(x)$ and defined as

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

Further,

let $N(C\varphi)$ and $R(C\varphi)$, respectively, denote the null space and therange space of $C\varphi$.

5. Composition operator on $\boldsymbol{\ell_p}$ spaces: Let φ be a self-map on the set of natural numbers $\mathbb{N}$.

Then φ induces a linear transformation $C\varphi$ on ℓp , defined by$x_n\chi n$

$$C\varphi\left(\sum_{n=1}^{\infty} x_n \chi n\right) = \sum_{\boldsymbol{n=1}}^{\infty} x_n \chi_{\varphi-1(n)}$$

where χ_n stands for the characteristic function of the set {n}.

In this section we discuss range and null spaces of composition operators on the Banach space also we discuss the injectivity and surjectivity of composition operator on the Banach space.

Theorem 5.1. A necessary and sufficient condition that a function φ on ℕinto itself induces a composition operator on ℓ_p $(1\leq p < \infty)$ is that

i. $|\varphi^{-1}(n)|$ is finite for each n in ℕ.

ii. The set $\{ \varphi^{-1}(n) : n \in \mathbb{N} \}$ is a bounded set.

Now we state a proposition about the norm of the Composition operator $C\varphi$ on ℓ_p $(1 \leq p < \infty)$

Theorem 5.2. Let $C\varphi$be a Composition operator on ℓ_p $(1 \leq p < \infty)$. Then $\| C\varphi \| = \text{Max}\ \{|\varphi^{-1}(n)|\}^{1/p}$, n ∈ ℕ.

Example 5.3. Define a self-map φ on N by

$$\varphi(n) = \begin{cases} n, & if\ n\ is\ odd \\ n-1, & if\ n\ is\ even \end{cases}$$

In this example we easy to see that $|\varphi^{-1}(n)| = 2$ for every n ∈ ℕ. Hence by using above theorem we get $\| C\varphi \| = \sqrt{2}$.

Example 5.4. Define φ on ℕ into itself by $\varphi(\boldsymbol{n}) = n + 1$ for each n ∈ ℕ

Then $C\varphi$ is a Composition operator on ℓ_p.

For $(x_1, x_2, x_3,\ldots, x_n,\ldots)$ in ℓ_p then $C\varphi(x_1,x_2,x_3,\ldots,x_n,\ldots) = (x_2,x_3, x_4, \ldots, x_{n+1},\ldots)$.

Thus $\| C\varphi \| =1$ and $C\varphi$ is the adjoint of a well known operator, the unilateral shift.

Theorem 5.5. Let $C\varphi$ be a composition operator on ℓ_∞ induced by function φ on ℕ into itself. Then $\| C\varphi \| = 1$.

Proof. Given that $C\varphi: \ell_\infty \to \ell_\infty$, then $(C\varphi f)(n) = f(\varphi(n))\ \forall\ n \in \mathbb{N}$.

Therefore,

$|(C\varphi f)(n)| = |f(\varphi(n))|\ \forall\ n \in N$

$\| C\varphi f \|= \sup |(f \circ \varphi)(n)|\ \forall n \geq 1.$

This implies that

$\| C\varphi f \|= \sup |f(\varphi(n))|\ \forall n \geq 1$

$\leq \sup n\geq1\ |f(n)|\ \forall n \geq 1$

$= \|f\|.$

Hence

$\| C\varphi f \| \leq \| f \|.$ …(1)

Conversely, let m ∈ Range(φ) then there exists n ∈ℕ such that φ(n) = m and n ∈ φ−1 (m). Let f = χ_m, then ‖f‖ = 1, and Cφf = Cφχm = χφ−1(m).

Then ‖ Cφf ‖ = ‖χφ−1(m)‖ = 1 this implies that ‖ Cφ ‖ ≥ 1. …(2)

From (1) and (2) we get ‖ Cφ ‖ =1.

Theorem 5.6. Let φbe a function on ℕinto itself such that Cφf ∈ ℓ_p for each f in ℓ_p. Then Cφ is a bounded linear operator on ℓ_p.

Now we discuss the null space and range space of Composition operator Cφon ℓ_p and discuss the range of Composition operator always closed, we further determine condition on φsuch that Composition operator injectivity and surjectivity .

Theorem 5.7. Let Cφ be a Composition operator on ℓ_p induced by a function φ in ℕinto itself. Then null space N(Cφ) of Cφ is given by

N(Cφ) = { f ∈ ℓ_p : f | φ(ℕ) = 0 }.

Theorem 5.8. Let Cφ be a Composition operator on ℓ_p induced by a function φ in ℕ into itself. Then range space R(Cφ) is given by

R(Cφ) = { f ∈ ℓ_p : f | φ^{-1}(n) = constant, ∀n ∈ ℕ }.

Theorem 5.9. Range space a Composition operator on ℓ_p is always closed.

Theorem 5.10. A Composition operator Cφon ℓ_p is one-one if and only if φ is onto.

Theorem 5.11. A Composition operator Cφ on ℓ_p is onto if and only if φ is one-one.

The following corollary is immediate consequence of the above theorems.

Corollary 5.12. A Composition operator Cφ on $\ell_p$is invertible if and only if φ is invertible.

6. **Sum of two Composition operator on ℓ_p spaces:** In this section we discuss a necessary and sufficient condition when range spaces of two Composition operators on the Banach space ℓp are equal. We also discuss a necessary and sufficient condition when null spaces of two Composition operators are equal.

Theorem 6.1. Let Cφ and Cψ be Composition operators on ℓp induced by functions φ and ψ on ℕ into itself. Then R(Cφ) ⊆ R(Cψ) if and only if for each n ∈ N, and for each p ∈ φ−1(n) there exists a natural number m such that

$$p \in \psi - 1(m) \subseteq \varphi - 1(n).$$

Theorem 6.2. Let $C\varphi$ and $C\psi$ be Composition operators on ℓp induced by functions φ and ψ on $\mathbb{N}$ into itself. Then $RC\varphi= R(C\psi)$ if and only if for each $n \in N$, there exists $m \in N$ such that $\psi-1(m) = \varphi-1(n)$.

Example 6.3. Define $\varphi, \psi : \mathbb{N} \to \mathbb{N}$ such that

$\varphi(1) = \varphi(2) = 1$, $\varphi(n) = n$, $n \geq 3$, and $\psi(2) = \psi(3) = 2$, $\psi(n) = n$, $n \notin \{2, 3\}$.

Then neither φ nor ψ satisfy the condition of above Hence $R(C\varphi) \subsetneq R(C\psi)$, and $R(C\psi) \subsetneq R(C\varphi)$.

Example 6.4. Define $\varphi, \psi : \mathbb{N} \to \mathbb{N}$ such that

$\varphi(1) = \varphi(2) =1$, $\varphi(n) = n$, $n \geq 3$, and$\psi(n) = n$ for each $n \in\mathbb{N}$.

Then by Theorem $R(C\varphi) \subset R(C\psi)$.

Theorem 6.5. Let $C\varphi$ and $C\psi$ be Composition operators on ℓp induced by functions φ and

ψ n $\mathbb{N}$ into itself. Then $N(C\varphi) = N(C\psi)$ if and only if $R(\varphi) = R(\psi)$.

Example 6.6. Define $\varphi, \psi : \mathbb{N} \to \mathbb{N}$ such that

$\varphi(1) = 2$, $\varphi(2) = 1$, $\varphi(n) = n$, $n \geq 3$, and $\psi(n) = n$ for each $n \in\mathbb{N}$.

Then, by $N(C\varphi) = N(C\psi)$.

Now we discuss that the sum of two Composition operators on ℓp cannot be a Composition operator on ℓp.

Theorem 6.6. Let $C\varphi$and $C\psi$ be Composition operators on ℓp induced by functions φ and ψ on $\mathbb{N}$ into itself. Then $C\varphi + C\psi$ can not be a Composition operator on ℓp.

Theorem 6.7. Let $C\varphi$ and $C\psi$ be Composition operators on ℓp induced by functions φ and ψ on $\mathbb{N}$ into itself. Then $C\varphi.C\psi$ be a Composition operator on ℓp.

Theorem 6.8. Let $C\varphi$ and $C\psi$ be Composition operators on ℓp induced by functions φ and ψ on $\mathbb{N}$ into itself. Then

$R(C\varphi + C\psi)\subset\{ f \in \ell p : f \mid \{\varphi-1(n)\cap\psi-1(m)\}\cup\{\varphi-1(m)\cap\psi-1(n)\}$ is constant $\forall$ $m, n \in\mathbb{N}\}$.

Theorem 6.9. Let $C\varphi$ and $C\psi$ be Composition operators on ℓp induced by functions φ and ψ on $\mathbb{N}$ into itself. Then null space $N(C\varphi + C\psi)$ of $C\varphi + C\psi$ is givenby

$N(C\varphi + C\psi) = \{f \in \ell p : f(\varphi(n)) + f(\psi(n)) = 0$ for each $n \in N\}$.

Theorem 6.10. Let $C\varphi$ and $C\psi$ be Composition operators on ℓp induced by functions φ and ψ on $\mathbb{N}$ into itself.If $C\varphi + C\psi$ is injective, then

(i).$R(\varphi) \cup R(\psi) =\mathbb{N}$

(ii).Either $\varphi-1(n) \neq \psi-1(m)$ or $\varphi-1(m) \neq \psi-1(n)$, whenever $m \neq n$.

Example 6.11. Define φ, ψ : ℕ → ℕ such that $\varphi(n) = n$, for each $n \in N$, and$\psi(1) = \psi(2) = 1$ and $\psi(n) = n$ for each $n \geq 3$.

$(C\varphi + C\psi)x = (2x_1, x_1 + x_2, 2x_3, ..., 2x_n, ...)$.

Hence $N(C\varphi + C\psi) = \{0\}$, thus $C\varphi + C\psi$ is injective in ℓp.

Cφ is injective (surjective) if and only if φ is surjective (injective) but this isnot the case

while dealing with sum of two Composition operators.

This is clear by the following example.

Example 6.12. Define φ, ψ:ℕ → ℕ such that $\varphi(n) = n$, for each $n \in ℕ$, and$\psi(1) = 2$, $\psi(2) = 1$ and $\psi(n) = n$ for each $n \geq 3$. Then φ and ψ are injective and surjective for $x \in \ell p$,

$(C\varphi + C\psi)(x) = (x_1 + x_2, x_2 + x_1, 2x_3, 2x_4, ...2x_n, ...)$.

From the above expression it is clear that $C\varphi + C\psi$ is neither surjective nor injective.

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