BOUNDS ON METRIC DIMENSION

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Abstract: In this paper, we present some bounds for metric dimension of a graph *G* in terms of order and some theoretic parameters such as diameter and maximum degree etc., Also, we characterize the Extremal graphs achieving the bounds.

Keywords: *Metric bounds, Distance partition, Metric dimension, Extremal graph*

1. INTRODUCTION

For any graph theoretic parameter, the study of determining bounds is the important one. Chartrand et. al [4] determined the bounds of the metric dimensions for any connected graphs and determine the metric dimension of some well known families of graphs such as paths and complete graphs. In [10], Khuller et. al considered graphs with small metric dimension and showed that a graph has metric dimension 1 if and only if it is a path and Chartrand et. al also proved this in [6]. Buczkowski et. al [1] proved the existence of a graph *G* with $\beta(G)=2$, for every integer $k \geq 2$. In this paper, we present some bounds for metric dimension of a graph *G* in terms of order and some theoretic parameters such as diameter and maximum degree etc.,

1.1. **Some Bounds for Metric dimension**

In this section, we determine some bounds for metric dimension and characterize the graph with metric dimension land *n - 1.* Also we characterize the extremal graphs achieving the bounds.

Theorem 1. 1. 1. If *G* is a graph on *n* vertices, then $1 \leq \beta(G) \leq n - 1$. For given integer *a* and *n* with $1 \leq a \leq n - 1$, there exists a graph *G* of order *n* such that $\beta(G)=a$.

Proof: The inequalities are trivial. Now suppose *a* and *n* are two integers with $1 \leq a \leq n - 1$. We construct a graph *G* of order *n* such that $\beta(G)=a$ as follows.

Case 1. *a = 1, 2, n- I, n- 2*

For *a = 1, 2, n — 1, n - 2,* let *G* be a graph with n vertices be taken as a path, cycle, complete graph and complete bipartite graph respectively. Then clearly $\beta(G)=a$.

Case 2. $3 \leq a \leq n-3$ and *n - a* is odd.

In this case, let G be a graph obtained from the cycle $3 \le a \le n-3 C_{n-a+1} = (v_1, v_2, ...v_{n-a+1}, v_1)$ by attaching *(a - 1)* pendent edges at any one of the vertices of the cycle say v_1 *and* let $x_1, x_2, ...x_{a-1}$ *be* the pendent vertices of *G*. We now claim that $\beta\ (G)$=a .

Let $S = (x_1, x_2, ...x_{a-2}, v_2, v_{n-a+1})$. It can be easily verified that S is a resolving set of G. So that $\beta(G) \le |S = a|$ .Next we have to show that $\beta\ (G) \ge$ a . For that , we have to prove the following Claim 1.

Claim 1. Every resolving set of *G* contains at least *a - 2* vertices from the set $X = \{x_1, x_2, ...x_{a-1}\}$.

Suppose not, then there exists a resolving set of *G* contains at most *a - 3* vertices say W_1 and so $|X - W_1| \ge 2$, However, if $x_i, x_j \in X - W_1$, then $\text{d}(x_i, v) = d(x_j, v), \forall v \in V(G)$. Hence no vertex of W_1 resolves x_i and x_j , a contradiction. This complete the proof of Claim 1.

Use the fact $\beta(C_n) = 2$ and Claim 1 we have $\beta\ (G) \ge$ a-2+2 and hence $\beta\ (G) = a$.

Case 3: $3 \le \text{a} \le \text{n} - 3$ and *n - a* is even.

Here also let *G* be the graph obtained from the cycle $C_{n-a+1} = (v_1, v_2, ...v_{n-a+1}, v_1)$ by attaching *(a- 1)* pendent edges at any one of the vertices of the cycle say v_j and attach one pendent edge at any other vertices of the cycle say v_1. Let $x_1, x_2, ...x_{a-1}$ be the pendent vertices of *G*, where x_a is incident with the pendent edge which is attached to v_2. We now claim that $\beta\ (G) = a$.

Let $S = (x_1, x_2, ...x_{a-2}, x_a, v_{n-a})$. Then it can be easily verified that *S* is a resolving set of *G* and so $\beta\ (G) \le |S| = a$. Next we have to prove that $\beta\ (G) \ge a$. For that, first we prove the following Claim 2.

Claim 2. Every resolving set of *G* contains at least 2 vertices from the set $T = C_{n-a} \cup \{x_a\}$.

Assume to the contrary, then there exists a resolving set of *G* contains at most one vertex from *T* say W_2. Note that if v_i and v_j are two distinct vertices of C_{n-a} with $\text{d}(v_i, v_1) = d(v_j, v_1)$ then $\text{d}(v_i, v') = d(v_j, v')$ for all $v' \in V(G) - C_{n-a} \cup \{x_a\}$, it follows that W_2 must contain exactly one vertex in *T*. We consider the following four cases.

Case (i). $x_a \in W_2$.

Since for any $v' \in V(G) - C_{n-a} \cup \{x_a\}$, $\text{d}(v_{n-a}, x_a) = d(v', x_a)$ and $\text{d}(v_{n-a}, u') = d(v', u')$ for any $u' \in V(G) - C_{n-a} \cup \{x_a\} \cup \{v'\}$ it follows that $r(v_{n-a} \setminus W_2) = r(v' \setminus W_2)$.

Case (ii). Any one of $\{v_1, v_2, ...v_{n/2-1}\}$ belongs to W_2.

Since for any $v' \in V(G) - C_{n-a} \cup \{x_a\} \cup \{v'\}$, $d(v_{n-a}, v_i) = d(v', v_i)$, $1 \le i \le n/2 - 1$ and $d(v_{n-a}, u') = d(v', u')$ for all $u' \in V(G) - C_{n-a} \cup \{v_i\} \cup \{x_a\}$, we have $r(v_{n-a} \setminus W_2) = r(v' \setminus W_2)$.

Case (iii). $v_{n/2} \in W_2$

Note that if v and v' are two distinct vertices of $C_{n\text{-}a}$ with $d(v,v_1)=d(v',v_1)$ then $d(v,v_{n/2})=d(v',v_{n/2})$ and $d(v, u) = d(v', u)$ for all $u\in V(G)-C_{n-a}\{x_a\}$ and so $r(v_2 \setminus W_2)=r(v' \setminus W_2)$

Case (iv). Any one of $\{v_{n/2}+1,....,v_{n-a}\}$ belongs to W_2.

Since for any $v' \in V(G)-C_{n-a}\cup\{x_a\}, d(v_2,v_i)=d(v',v_i)$, $n/2+1\le i\le n-a$ and $d(v_2,u')=d(v',u')$ for all $u' \in V(G)-C_{n-a}\cup\{v_i\}\cup\{x_a\}$ We have $r(v_2 \setminus W_2)=r(v' \setminus W_2)$.

In each case, W_2 is not a resolving set of *G*, a contradiction. Therefore, every resolving set of *G* contains at least two vertices from the set *T*. From Claim 1 and Claim 2 $\beta(G)\ge a$ and hence $\beta(G)=a$.

Illustration (i). If *n = 10 and a = 5*, then the required graph *G* is given Figure 1.1.1. This is actually discussed in Case 2. One can verify that $\beta(G)=5$.

Illustration(ii). If *n = 12* and *a = 4*, then the required graph *G* is given in Figure 1.1.2. This is actually discussed in Case 3. One can easily verify that $\beta(G)=4$.

In the Theorems 1.1.2. and 1.1.3., we characterize the extremal graphs achieving the bounds given in Theorem 1.1.1.

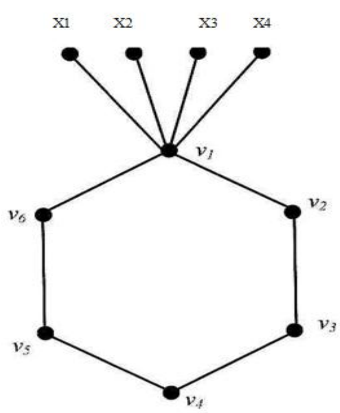
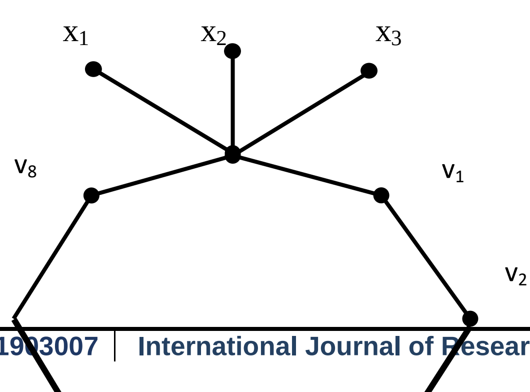


Figure 1.1.1



v_6

v_5 v_3

Figure 1.1. v_4

Theorem 1. 1. 2. A connected graph G of order n has metricdimension 1 if and only if $G \cong P_n$.

Proof: Let G be a graph with $\beta\ (G) = 1$. We have to prove that G is apath.

Let $W = \{\ w\ \}$ be a minimum resolving set for G. For each vertex $v \in V(G), r(v / W) = d(v, w)$ is a non negative integer less than n. Since the codes of the vertices of G with respect to *W* are distinct, there exists a vertex *u* of G such that *d(u, w) = n — 1*. Consequently, the diameter of G is *n - 1*. This implies that $G \cong P_n$.For the converse, let G be a path on *n* vertices. By Proposition 1.1.1, $\beta\ (G) = 1$.

Theorem 1.1.3. Let G connected graph of order $n \geq 2$. Then $\beta\ (G) = n-1$if and only if $G \cong K_n$.

Proof: Let G be a graph with $\beta\ (G) = n-1$. We will show that $G \cong K_n$. Suppose not. Then G contains two vertices *u* and v with *d(u, v) = 2*. Let *u, x, v* be a path of length 2 in G and let *W = V(G) - {x, v}*. Since *d(u, v) = 2* and *d(u, x) = 1*, it follows that *r (x \W) = r(v \W)* and so *W* is a resolving set. Which is contradiction to the fact that $\beta\ (G) = n-1$. For the converse, assume that $G \cong K_n$. By Proposition 1.1.12. $\beta\ (G) = n-1$. □

In the following theorem we determine some bounds for the metric dimension of a graph in terms of maximum degree and diameter.

Theorem 1.1.4. Let *G* be a nontrivial connected graph of order $n \geq 2$, diameter *d(G)*, and maximum degree $\Delta(G)$. Then

$$[\log_3(\Delta(G)+1)] \leq \beta(G) \leq n - d(G)\ .$$

Proof: First, we establish the upper bound. Let *u* and v be vertices of *G* for which *d(u, v) = d(G)* and let $u = v_0, v_1, v_2, ...v_{d(G)} = v$ be a shortest *u*-*v*path.

Let *W = V(G)- {* $v_1, v_2, ...v$ *}*. Since $u \in W$ and $d(u, v_i) = i$ for $1 \leq i \leq d(G)$, it follows that *W* is a resolving set of cardinality *n - d(G)* for *G*. Thus $\beta(G) \leq n - d(G)$.

Next, we consider the lower bound. Let $\beta(G)=k$ and let $v\in V(G)$ with $\deg v=\Delta$. Moreover, let N(v) be the neighbourhood of v and let $W=\{w_1,w_2,.....w_k\}$ be a resolving set of *G.* Observe that if *u e N(v),* then for each $1\le i\le k$, the distance $d(u,w_i)$ is one of the numbers $d(v,w_i), d(v,w_i)+1$ or $d(v,w_i)-1$. Moreover, since *W* is a resolving set, *r(u \ W)= r(v \ W)* for all $u\in N(v)$. Thus there are three possible number for each of the *k* coordinates of *r(u \ W).* On the otherhand, since it cannot occur that *d(u,w$_i$)* = *d(v,w$_i$)* for all *i* ($1\le i\le k$), it follows that there at most *3^k - 1* distinct codes of the vertices in *N(v)* with respect to *W.* Therefore, $|N(v)|=\Delta\le 3^k-1$, which implies that $\beta\ (G)=k\ge \log_3(\Delta(G)+1)$. Since /?(G) is an integer, $\beta\ (G)\ge \log_3(\Delta(G)+1)$.

1.2. Graphs with β = n-2

This section completely characterizes the family of graphs of order *n* for which the metric dimension *n-2.*

Theorem **1.2.1.** Let G be a connected graph of order $n\ge 4$. Then $\beta=$ n-2 if and only if $G=K_{s,t}(s,t\ge 1), G=K_s+\overline{K_t}, (s\ge 1, t\ge 2),$ *or* $\mathrm{G}=\mathrm{K}_s+(K_l\cup K_t)(s,t\ge 1)$

Proof: It can be easily show that $\beta(G)\le$ n-2 for each of the graphs mentioned in the statement of the theorem. To see $\beta(G)\ge$ n-2, note that if the vertices of a graph are partitioned as $V_1\cup V_2\cup.....\cup V_p$ where either *V$_i$*is independent and its vertices have identical open neighborhoods or *V$_i$*induces a clique and its vertices have identical closed neighborhoods, then the metric dimension is at least $(|V_1|-1)+(|V_2-1|).......+(|V_p|-1)$. Since each of the graphs mentioned in the statement of the theorem are partition as $V_1\cup V_2$,then the metric dimension is at least $(|V_1|-1)+(|V_2-1|)$. Therefore $\beta(G)\ge$ n-2 and hence $\beta(G)=$ n-2.

For the converse, assume that *G* is a connected graph of order $n\ge 4$ such that $\beta(G)=$ n-2. By Theorem 1.1.4. and, it follows that *G* has diameter 2. If *G* is bipartite and since the diameter of *G* is 2, $G=K_{s,t}$ for some integers *s, t > 1.*

Hence, we may assume that G is not bipartite. Therefore, G contains an odd cycle. Let C_r be a smallest odd cycle in G. We claim that $r = 3$. Certainly, $C_r$is an induced cycle of G. If G contains an induced &-cycle $v_}, v_2,..., v_k$, where k >5, then $W = V(G) - \{\>2>$vj, $v_4\}$ is a resolving set of cardinality $n - 3$, for if we let $wi = V]$ and $w_2 = v_5$, then $r(v_2 \setminus W) = (1, s, . . .)$, $r(v_3 \setminus W) = (2, 2, ...)$ and $r(v_4 \setminus W) = (t, 1, . . .)$ where $s, t > 2$. Hence, $p (G) < n - 3$, which is a contradiction. Thus G has no induced cycle of length $k > 5$ and so $r = 3$ and G contains a triangle.

Let Y be the vertex set of a maximum clique of G. Since G contains a triangle, $\setminus Y \setminus > 3$. Let $U \sim V(G) - Y$. Since G is not complete, $\setminus U \setminus > 1$. $If \setminus U \setminus = 1$, then $G = K_s + (K_}KJKJ$for some integers s and t. Now, $s > 1$ since G is connected and t >1 since G is not complete. From these observations, we may assume that $\setminus U \setminus > 2$.

First, we show that t/is an independent set of vertices. Suppose, to the contrary, that U is not independent. Then U contains two adjacent vertices u and w. Because of the defining property of Y, there exists v e Y such that uv t $E(G)$ and v' e Y such that wv' g $E(G)$, where v and v' are not necessarily distinct. We consider the following two cases.

Case 1. There exists a vertex v e Y such that uv, wvg $E(G)$.

The following two cases are to be discussed.

Subcase 1.1. There exists a vertex x e 7that is adjacent to exactly one of u and w, say u.

(b)

Since | $Y \setminus > 3$, there exists a vertex y e Y that is distinct from v and x. Thus G contains the subgraph shown in Figure 1.2.1 (a), where dotted lines indicate that the given edge is not present.

Let $W = V(G) - \{u, w, yj$. Letting w; = v and $w_2 = x_t$we have

$r(u\setminus W) = (2, 1, \ldots)$,

$r(w\setminus W) = (2,2,...)$,

$r(y\setminus W) = (1, 1, \ldots)$. So $W$is a resolving set of cardinality $n - 3$, which is a contradiction.

Subcase 1.2. Every vertex of Fis adjacent to either both u and w or to neither$u$ nor w.

If *u* and *w*are adjacent to every vertex in *Y* - *{v}*, then the vertices of (F-*{v})* u *{u, w}* are pair wise adjacent, contradicting the defining property of *Y*. Thus, there exists a vertex v e *Y* such that *y* is distinct from v, and *y* is adjacent to neither *u* nor *w*.

Since the diameter of *G* is 2, there is a vertex x of G that is adjacent to both *u* and v. Thus *G* contains the subgraph shown in Figure 3.2.1 (b), where dotted lines indicate that the given edges are not in *G*.

Let *W* - *V(G)* - *{x, y, w}* and label w_t— *v* and w_2— *u*. Then

$r(x\backslash W) = (1,1,\dots)$,

$r(y\backslash W) = (1,2,\dots)$,

$r(w\backslash W) = (2,1,\dots)$.

Thus *W* is a resolving set of cardinality *n - 3,* producing a contradiction.

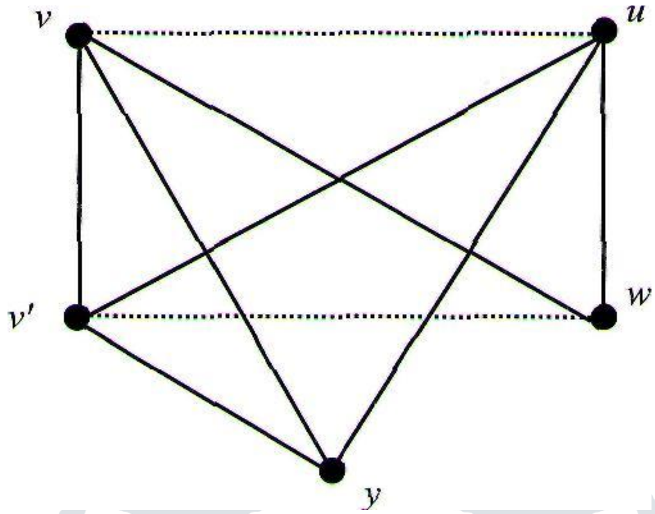
Case 2. For each vertex v of *Y*, *v* is adjacent to at least one of *u* and *w*.

Because *Y* is the vertex set of a maximum clique, there exist vertices v, v' e *Y* such that *uv, wv' e E(G)*. Necessarily, *vw, v'u*e *E(G)*. Since | *Y* \ > *3,* there exists a vertex *y* in *Y* distinct from *v* and v'. Now, at least one of the edges *yu*and *yw*must be present in *G*, say *yu*. Thus, G contains the subgraph shown in Figure 3.2.2 (a) where again dotted edges indicate that the given edge is not in G.

Let *W* = *V(G)* - *{u, w, y}*and label in *wj*= v and w_2 = v'. Then

$r(u|W) = (2,l,\dots), r(w\backslash W) = (1,2,\dots), r(y\backslash W) = (1,1,\dots)$.

Again, *W* is a resolving set of cardinality *n — 3*, which is a contradiction. Thus, as claimed, *U* is independent.



(a)

(b)

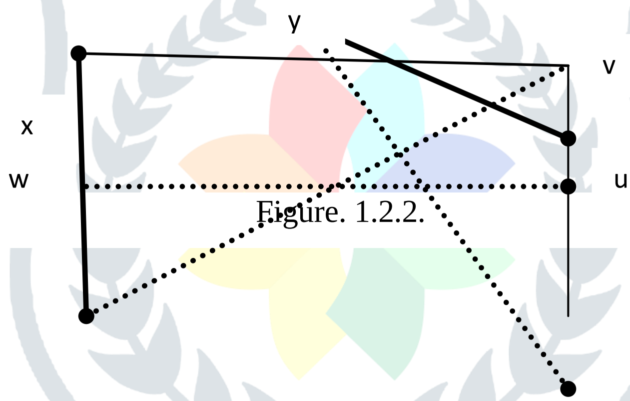


Figure. 1.2.2.

Next, we claim that $N(u) = N(w)$ for all u, w e U. Let u and w be two vertices of U. Suppose that uv e $E(G)$ for some vertex v of G. Necessarily, v e Y. We show that wve $E(G)$. Assume, to the contrary, that wv€ $E(G)$. Since G is connected and U is independent, w is adjacent to some vertex of Y. If w is adjacent only to y, then since $w$and y are not adjacent to u, $d(w, u) = 3$, which contradicts the fact that the diameter of G is 2. Thus there exists a vertex x in 7 distinct from y such that wxe $E(G)$. Therefore, G contains the subgraph shown in Figure 3.2.2 (b), where again dotted edges are not in G.

Let $W = V(G) - \{u, w, x\}$ and label wj = v and $w_2 = y$. Then

$r(u\backslash W) = (l,2,.,.)$,

$r(w\backslash W) = (2,...)$,

$r(x\backslash W) = (l,l,...)$.

Thus, W is a resolving set of cardinality $n - 3$, producing a contradiction.

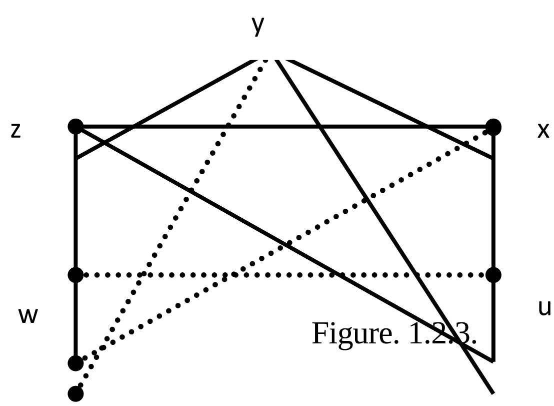


Figure. 1.2.3.

Therefore $V(G) = Y \cup U$, where Y induces a clique, U is independent, $\backslash Y \backslash \geq 3$, $\backslash U \backslash \geq 2$, and $N(u) = N(w)$ for all $u, w \in U$.

Next, we claim that for u € U, there is at most one vertex of Y not contained in $N(u)$. Suppose, to the contrary, that there are two vertices $x, y \in Y$not in $N(u)$. Let W be a vertex of U that is distinct from u. Therefore, $N(w) = N(u)$. Since G is connected, there exists $z \in$ 7 such that z $\in$ N(u) = $N(w)$. Thus G contains the subgraph shown in Figure 1.2.3., where dotted edges are not edges of G.

Let $W = V(G) - \{y, w, z\}$ and label wj= x and w_2= u. Then

$r(y\backslash W) = (1,2,\ldots)$,

$r(w\backslash W) = (2,2,\ldots)$,

$r(z \backslash W) = (1, 1, \ldots)$.

Hence, W is a resolving set of cardinality $n - 3$, producing a contradiction.

Now, $N(u)$=Y For $N(u)=Y-\{v\}$ for some v $\in$ 7. If $N(u) = Y$, *then* $G=K_S+ K_t$*for* $s = \backslash Y \backslash \geq 3$ *a n d* $t = \backslash U \backslash \geq 2$. If $N(u) = Y- \{v\}$, then $G = K_s + \left(K_1 \cup K_{\bar{t}}\right)$,

where $V(K_1) = \{v\}$, S = | Y | - 1 $\geq$ 2, and T = | U | ≥ 2.

However, $K_s + \left(K_1 \cup K_{\bar{t}}\right) = K_s + K_{\overline{t+1}}$. In either case, G is the join of a complete graph and an empty graph.

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