ON ROUGH IDEALS IN Γ-NEAR-RINGS

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Abstract

The aim of this paper is to present the concepts of congruence relation in Γ-near-rings and the lower and upper approximations of an ideal with respect to the congruence relation and introduce the notion of rough ideals in Γ-near-rings, which is a generalization of the notion of near rings. Some properties of the lower and upper approximations are discussed in Γ-near-rings.

Keywords: Γ-near-rings, Congruence relation, Rough ideals.

1 Introduction

Γ-near-ring and the ideal theory of Γ-near-ring were introduced by Bh. Sathyanaranan[7]. For basic terminology in near-ring we refer to Pilz[6] and in Γ-near-ring.

Pawlak [3-5]introduced the theory of rough sets in 1982. It is an another independent method to deal the vagueness and uncertainty. Pawlak used equivalence class in a set as the building blocks for the construction of lower and upper approximations of a set. Many researchers studied the algebraic approach of rough sets in different algebraic structures such as [1,2,8,9]. Thillaigovindan and Subha[10] introduces rough ideals in near-rings.

The aim of this paper is to present the concepts of congruence relation in Γ-near-rings and the lower and upper approximations of an ideal with respect to the congruence relation and introduce the notion of rough ideals in Γ-near-rings, which is a generalization of the notion of near rings. Some properties of the lower and upper approximations are discussed in Γ-near-rings.

2 Preliminaries and Congruence Relation

We first recall some basic concepts for the sake of completeness. Recall from[], that a non empty set N with two binary operations $+$ and $\bullet$ multiplication is called a near-ring, if it satisfies the following axioms.
(i) $(N,+)$ is a group; (ii) $(N,\bullet)$ is a semigroup; (iii) $(n_1+n_2)\bullet n_3 = n_1\bullet n_3 + n_2\bullet n_3$, for all $n_1, n_2, n_3 \in N$.

Definition 2.1. [7] A Γ-near-ring is a triple where $(\mathrm{M},+,\Gamma)$ where

i) $(\mathrm{M},+)$ is a group
ii) Γ is non empty set of binary operators on M such that for $\alpha \in \Gamma$, $(\mathrm{M},+,\alpha)$ is a near-ring
iii) $x\alpha(y\beta z) = (x\alpha y)\beta z$ for all $x,y,z \in \mathrm{M}$ and $\alpha,\beta \in \Gamma$.

In Γ-near-ring, $0\gamma x = 0$ and $(-x)\gamma y = -x\gamma y$, but in general $x\gamma 0 \neq 0$ for some $x \in \mathrm{M}, \gamma \in \Gamma$. More precisely the above near-ring is right near-ring.

$M_0 = \{n \in M \,/\, n\gamma 0 = 0\}$ is called the zero-symmetric part of M and
$\mathrm{M} = \{n \in \mathrm{M}/\, n\gamma 0 = n, for\ all\ \gamma \in \Gamma\} = \{n \in \mathrm{M}/\, n\gamma n' = n\ for\ all\ n' \in \mathrm{M}, \gamma \in \Gamma\}$ is called the constant part of M. M is called *zero-symmetric* if $\mathrm{M} = \mathrm{M}_0$ and M is called constant if $\mathrm{M} = \mathrm{M}_c$.

Definition 2.2. A subset I of a Γ-near-ring M is called a left*(resp. right) ideal* of M , if

i) $(I,+)$ is a normal divisor of $(\mathrm{M},+)$ and
ii) $a\alpha(x+b) - a\alpha b \in I$ (resp. $x\gamma b \in I$) for all $x \in I, \alpha \in \Gamma$ and $a,b \in$ M.

Let I be an ideal of M and X be a non-empty subset of M. Then the sets $\underline{\rho}_{\mathrm{I}}(X) = \{x \in \mathrm{M}/\, \mathrm{x}+\mathrm{I} \subseteq X\}$ and $\overline{\rho}_{\mathrm{I}}(X) = \{x \in M/\, (\mathrm{x}+\mathrm{I}) \cap \mathrm{X} \neq \emptyset\}$ are called respectively the lower and upper approximations of the set X with respect to the ideal I.

For any ideal I of M and $a,b \in \mathrm{M}$, we say a is congruent to b mod I, written as $a \equiv b(mod\ A)$ if $a-b \in I$.

It is easy to see that relation $a \equiv b(mod\ A)$ is an equivalence relation. Therefore, when $U = \mathrm{M}$ and θ is the above equivalence relation, we use the air (M,A) instead of the approximation space (U,θ).

Also, in this case we use the symbols $\underline{\rho}_I(X)$ and $\overline{\rho}_I(X)$ instead of $\underline{\rho}(X)$ and $\overline{\rho}(X)$. If X is a subset of M, then X^C will be denoted by $M - X$.

3. SOME PROPERTIES OF ROUGH APPROXIMATIONS

In this section we study some fundamental properties of the lower and upper approximations of any subsets of a Γ-near-ring with respect to an ideal. Throughout this paper M denotes the Γ-near-ring unless otherwise specified.

Lemma 3.1. For every approximation space (M, I) and every subsets $X, Y \subseteq M$, the following hold:

1) $\underline{\rho}_I(M - X) = M - \overline{\rho}_I(X)$
2) $\overline{\rho}_I(M - X) = M - \underline{\rho}_I(X)$
3) $\overline{\rho}_I(X) = (\underline{\rho}_I(X^C))^C$
4) $\underline{\rho}_I(X) = \left(\overline{\rho}_I(X^C)\right)^C$.

Proof. Straight forward.

Theorem 3.2. *For every approximation space* (M, I) *and ever subsets* $X, Y \subseteq M$, *then the following hold:*

1) $\underline{\rho}_I(X) \subseteq X \subseteq \overline{\rho}_I(X)$
2) $\underline{\rho}_I(\emptyset) = \emptyset = \overline{\rho}_I(\emptyset)$
3) $\underline{\rho}_I(M) \subseteq M \subseteq \overline{\rho}_I(M)$
4) $\overline{\rho}_I(X \cup Y) = \overline{\rho}_I(X) \cup \overline{\rho}_I(Y)$
5) $\underline{\rho}_I(X \cap Y) = \underline{\rho}_I(X) \cap \underline{\rho}_I(Y)$
6) If $X \subseteq Y$, then $\underline{\rho}_I(X) \subseteq \overline{\rho}_I(Y)$ and $\overline{\rho}_I(X) \subseteq \overline{\rho}_I(Y)$
7) $\overline{\rho}_I(X \cap Y) \subseteq \overline{\rho}_I(X) \cap \overline{\rho}_I(Y)$
8) $\underline{\rho}_I(X \cup Y) \supseteq \underline{\rho}_I(X) \cap \underline{\rho}_I(Y)$
9) If J is an ideal of M such that $I \subseteq J$, then $\underline{\rho}_I(A) \supseteq \underline{\rho}_J(A)$ and $\overline{\rho}_I(A) \subseteq \overline{\rho}_J(A)$
10) $\underline{\rho}_I(\underline{\rho}_I(X)) = \underline{\rho}_I(X)$
11) $\overline{\rho}_I(\overline{\rho}_I(M)) = \overline{\rho}_I(X)$
12) $\overline{\rho}_I(\underline{\rho}_I(M)) = \underline{\rho}_I(X)$
13) $\underline{\rho}_I(\overline{\rho}_I(M)) = \overline{\rho}_I(X)$
14) $\underline{\rho}_I(x + I) = \overline{\rho}_I(x + I)$ for all $x \in M$.
15) $\underline{\rho}_I(X)\Gamma\underline{\rho}_I(Y) \subseteq \underline{\rho}_I(X\Gamma Y)$
16) $\overline{\rho}_I(X)\Gamma\overline{\rho}_I(Y) \subseteq \overline{\rho}_I(X\Gamma Y)$.

Corollary 3.4. Let (M, I) be any approximation space. Then

(i) For every $A \subseteq M$, $\underline{\rho}_I(A)$ *and* $\overline{\rho}_I(A)$ are definable sets

(ii) For every $x \in M$, $x + I$ is definable set.

Theorem 3.5. Let I *be an ideal of* M *and* A, B *nonempty subsets of* M, *then*
$\overline{\rho}_I(A)\Gamma\overline{\rho}_I(B) = \overline{\rho}_I(A\Gamma B)$.

Proof. Let $x \in \overline{\rho}_I(A)\Gamma\overline{\rho}_I(B)$. Then $x = a\gamma b$ for some $a \in \overline{\rho}_I(A)$ and $a \in \overline{\rho}_I(B)$. There exist $y, z \in M, \gamma \in \Gamma$ such that $y \in (a + I) \cap A$ and $z \in (b + I) \cap B$. Hence $y\gamma z \in A\Gamma B$ and
$y\gamma z \in (a + I)\Gamma(b + I)$. This implies that $y\gamma z \in a\gamma b + I = x + I$ and hence $a \in \overline{\rho}_I(A\Gamma B)$. Hence

$$\overline{\rho}_I(A)\Gamma\overline{\rho}_I(B) \subseteq \overline{\rho}_I(A\Gamma B). \quad (1)$$

On the other hand assume that $x \in \overline{\rho}_I(A\Gamma B)$. Then there exists $y \in M$ such that $y \in x + I$ and $y \in A\Gamma B$. This implies that $y = a_1\gamma b_1$ for some $a_1 \in A$ and $b_1 \in B$. Since
$x \in y + I = a_1\gamma b_1 + I = (a_1 + I)\Gamma(b_1 + I)$, x can be expressed as $x = x_1\gamma x_2$ for some

$x_1 \in a_1 + I$ and $x_2 \in b_1 + I$. This implies that $a_1 \in x_1 + I$ and $b_1 \in x_2 + I$ and so $y = a_1\gamma b_1$ and $(x_1 + I) \cap A \neq \emptyset$ and $(x_2 + I) \cap B \neq \emptyset$. This means that $x_1 \in \overline{\rho}_I(A)$ and $x_2 \in \overline{\rho}_I(B)$. Thus $x = x_1\gamma x_2 \in \overline{\rho}_I(A)\Gamma\,\overline{\rho}_I(B)$ and hence

$\overline{\rho}_I(A\Gamma B) \subseteq \overline{\rho}_I(A)\Gamma\,\overline{\rho}_I(B)$ (2)

Combining (1) and (2), we obtain $\overline{\rho}_I(A\Gamma B) = \overline{\rho}_I(A)\Gamma\,\overline{\rho}_I(B)$.

Theorem 3.6. Let I be an ideal of M and A, B nonempty subsets of Ň, then
$\underline{\rho}_I(A)\Gamma\underline{\rho}_I(B) \subseteq \underline{\rho}_I(A\Gamma B)$.

Proof. Let $x \in M$. Suppose $x \in \underline{\rho}_I(A)\Gamma\underline{\rho}_I(B)$. Then $x = a\gamma b$ for some $a \in \underline{\rho}_I(A)$ and $b \in \underline{\rho}_I(B)$. Hence $a + I \subseteq A$ and $b + I \subseteq B$. Now $(a + I)\Gamma(b + I) \subseteq A\Gamma B$ and $a\gamma b + I \subseteq A\Gamma\, B$. This implies that $x + I \subseteq A\Gamma B$. Hence $x \in \underline{\rho}_I(A\Gamma\, B)$.
Thus $\underline{\rho}_I(A)\Gamma\underline{\rho}_I(B) \subseteq \underline{\rho}_I(A\Gamma B)$.

Example 3.7. Let $N = \{0, a, b, c\}$ and $\Gamma = \{0, a, b\}$. Define addition and multiplication in M as follows:

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Γ	0	a	b
0	0	0	0
a	0	a	a
b	0	b	b

Then $(M, +, \Gamma)$ is a Γ-near-ring
Let $I = \{0, a\}, A = \{b, c\}, B = \{a, b\}$. Then $\underline{\rho}_I(A) = \{x \in M \,/\, x + I \subseteq A\} = \{b, c\}$.
$\underline{\rho}_I(B) = \{x \in M \,/\, x + I \subseteq B\} = \emptyset$. $\underline{\rho}_I(A)\Gamma\underline{\rho}_I(B) = \emptyset$. $\underline{\rho}_I(A\Gamma\, B) = \{b\}$.
Then $\underline{\rho}_I(A\Gamma B) \nsubseteq \underline{\rho}_I(A)\Gamma\,\underline{\rho}_I(B)$.

Theorem 3.8. *Let I be an ideal of M and A, B nonempty subsets of M, then*
$\overline{\rho}_I(A) + \overline{\rho}_I(B) = \overline{\rho}_I(A + B)$.

Proof. Let $x \in \overline{\rho}_I(A) + \overline{\rho}_I(B)$. Then $x = a + b$ for some $a \in \overline{\rho}_I(A)$ and $b \in \overline{\rho}_I(B)$. there exist $y, z \in M$ such that $y \in (a + I) \cap A$ and $z \in (b + I) \cap B$.
Now $y + z \in A + B$ and $y + z \in (a + I) + (b + I) = (a + b) + I = x + I$. This shows that $x + I \cap A + B \neq \emptyset$ and hence $x \in \overline{\rho}_I(A + B)$. Thus

$\overline{\rho}_I(A) + \overline{\rho}_I(B) \subseteq \overline{\rho}_I(A + B)$ (3)

Conversely, assume that $x \in \overline{\rho}_I(A + B)$. There exists $y \in M$ such that $y \in x + I$ and $y \in A + B$. This implies $y = a_1 + b_1$ for some $a_1 \in A$ and $b_1 \in B$. Since
$x \in y + I = (a_1 + b_1) + I = (a_1 + I) + (a_1 + I)$, x can be expressed as $x = x_1 + x_2$ for some $x_1 \in a_1 + I$ and $x_2 \in b_1 + I$. This means that $a_1 \in x_1 + I$ and $b_1 \in x_2 + I$ and hence $y = a_1 + b_1$ and $(x_1 + I) \cap A \neq \emptyset$ and $(x_2 + I) \cap B \neq \emptyset$. This means that $x_1 \in \overline{\rho}_I(A)$ and $x_2 \in \overline{\rho}_I(B)$. Thus $x = x_1 + x_2 \in \overline{\rho}_I(A) + \overline{\rho}_I(B)$.

Hence $\overline{\rho}_I(A + B) \subseteq \overline{\rho}_I(A) + \overline{\rho}_I(B)$ (4)

Combining (3) and (4), we obtain $\overline{\rho}_I(A + B) = \overline{\rho}_I(A) + \overline{\rho}_I(B)$.

Theorem 3.9. *Let I be an ideal of M and A, B nonempty subsets of* M, *then*
$\underline{\rho}_I(A) + \underline{\rho}_I(B) \subseteq \underline{\rho}_I(A + B)$.

Proof. Let $x \in$ M. Suppose $x \in \underline{\rho}_I(A) + \underline{\rho}_I(B)$. Then $x = a + b$ for some $a \in \underline{\rho}_I(A)$ and
$b \in \underline{\rho}_I(B)$. Hence $a + I \subseteq A$ and $b + I \subseteq B$.Now $(a + I) + (b + I) \subseteq A + B$ and
$(a + I) + I \subseteq A + B$. This implies that $x + I \subseteq A + B$. Hence $x \in \underline{\rho}_I(A + B)$, and thus
$\underline{\rho}_I(A) + \underline{\rho}_I(B) \subseteq \underline{\rho}_I(A + B)$.

The reverse inclusion of the Theorem 3.9 is not true in general which is shown in the following example.

Example 3.10. Consider the same example as in Example 3.7,

$$\underline{\rho_I}(A+B) = \{0,a,b,c\};$$

$$\underline{\rho_I}(A) = \{b,c\}, \underline{\rho_I}(B) = \emptyset;\ \underline{\rho_I}(A) + \underline{\rho_I}(B) = \{b,c\}.$$

Hence $\underline{\rho_I}(A+B) \nsubseteq \underline{\rho_I}(A) + \underline{\rho_I}(B)$.

Lemma 3.11. *Let I, J be two ideals of M and A a nonempty subset of M, then*

(i) $\underline{\rho_I}(A) \cap \underline{\rho_J}(A) \subseteq \underline{\rho_{I\cap J}}(A)$

(ii) $\overline{\rho}_{I\cap J}(A) \subseteq \overline{\rho}_I(A) \cap \overline{\rho}_J(A)$.

Proof. (i) Since $I \cap J \subseteq I, J$ by Theorem 3.2(9) we have, $\underline{\rho_I}(A) \subseteq \underline{\rho_{I\cap J}}(A)$ and $\underline{\rho_J}(A) \subseteq \underline{\rho_{I\cap J}}(A)$. Hence $\underline{\rho_I}(A) \cap \underline{\rho_J}(A) \subseteq \underline{\rho_{I\cap J}}(A)$.

Again $I \cap J \subseteq I, J$ we have, $\overline{\rho}_{I\cap J}(A) \subseteq \overline{\rho}_I(A)$ and $\overline{\rho}_{I\cap J}(A) \subseteq \overline{\rho}_{I\cap J}(A)$.

Hence$\overline{\rho}_{I\cap J}(A) \subseteq \overline{\rho}_I(A) \cap \overline{\rho}_J(A)$.

This reverse inclusions of Lemma 3.11 are not true in general which is shown in the following example.

Example 3.12. Let $M = \{0,a,b,c\}$ and $\Gamma = \{0,a,b,\}$ Define addition and Γ in M as follows:

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Γ	0	a	b
0	0	0	0
a	0	a	0
b	0	0	b

Then $(M,+,\Gamma)$ is a Γ-near-ring.

Let $I = \{0,a\}, J = \{0,b\}$ and $A = \{0,a,c\}$. Then I and J are ideals of M. $\underline{\rho_I}(A) = \{0,a\}$, $\underline{\rho_J}(A) = \{a,c\}, \overline{\rho}_I(A) = M, \overline{\rho}_J(A) = M$ and $\overline{\rho}_{I\cap J}(A) = \{0,a,b\}$, $\underline{\rho_I}(A) \cap \underline{\rho_J}(A) = \{a\}, \underline{\rho_{I\cap J}}(A) = A$.

$\underline{\rho_I}(A) \cap \underline{\rho_J}(A) \nsupseteq \underline{\rho_{I\cap J}}(A)$ and $\overline{\rho}_{I\cap J}(A) \nsupseteq \overline{\rho}_I(A) \cap \overline{\rho}_J(A)$.

Theorem 3.13. *If I and J are two ideals (resp. sub near-rings) of M, then $\overline{\rho}_I(J)$ is an ideal (resp. sub near-ring) of M.*

Proof. Let I and J be ideals of M and $i, j \in \overline{\rho}_I(J)$. Then there exist $p \in (i+I) \cap J$ and $q \in (j+I) \cap J$. Since J is an ideal of M, $p - q \in J$,

$$\begin{aligned} p - q \in (i+I) - (j+I) &= i + I + I - j \\ &\subseteq i + I - j \\ &= i - j + (j + I - j) \\ &\subseteq i - j + I. \end{aligned}$$

This implies that $((i-j) + J \cap J \neq \emptyset$ and so $i, j \in \overline{\rho}_I(J)$.

Assume that $x \in \overline{\rho}_I(J)$ and $a \in M$. Then there exists $p \in (x+I) \cap J$ such that $p \in x + I$ and $p \in J$. Since J is an ideal of Ň, $a + p - a \in J$ and

$$\begin{aligned} a + p - a \in a + x + I - a &= a + x - a + a + I - a \\ &\subseteq a + x - a + I. \end{aligned}$$

This shows that $(a + x - a + I) \cap J \neq \emptyset$ and $a + x - a \in \overline{\rho}_I(J)$.

Suppose $p \in \overline{\rho}_I(J)$ and $a \in M$. There exists $j \in M$ such that $j \in (p+I) \cap J$. J being an ideal of M, $j\gamma a \in J$ and $j\gamma a \in (p+I)\gamma a = p\gamma a + I$. Thus $(p\gamma a + I) \cap J \neq \emptyset$ and $p \in \overline{\rho}_I(J)$. Hence $\overline{\rho}_I(J)\Gamma$Ň $\subseteq \overline{\rho}_I(J)$. Let $a, b \in M$ and $p \in \overline{\rho}_I(J)$. So there exists $i \in (p+I) \cap J$. Since J is an ideal of M, $a\gamma(b+i) - a\gamma b \in J$ and

$$a\gamma(b+i)-a\gamma b \in a\gamma\big(b+(p+I)\big)-a\gamma b \subseteq a\gamma(b+p)-a\gamma b+I.$$

Thus $\{(a\gamma(b+p)-a\gamma b)+I\}\cap J \neq \emptyset$ and hence $a\gamma(b+p)-a\gamma b \in \overline{\rho}_I(J)$. Thus $\overline{\rho}_I(J)$ is an ideal of M.

Theorem 3.14. If I and J are two ideals (resp. sub Γ-near-rings) of M, then $\underline{\rho_I}(J)$ is an ideal (resp. sub Γ-near-ring) of M.

Proof. Let I and J be two ideals of M. Let $x, y \in \underline{\rho_I}(J)$. Then $x+I, y+I \subseteq J$. Since J is an ideal of M, $(x+I)-(y+I) \subseteq J$ and so $x-y+I \subseteq J$. Hence $x-y \in \underline{\rho_I}(J)$. Assume that $x \in \underline{\rho_I}(J)$ and $a \in M$. This implies that $x+I \subseteq J$ and J being an ideal of
M, $a+(x+I)-a \subseteq J$ and $a+(x+I)-a \subseteq \underline{\rho_I}(J)$. Let $x \in \underline{\rho_I}(J)$ and $a \in M$. Then $x+I \subseteq J$ and $(x+I)\gamma a \subseteq J$. Hence $x\gamma a+I=(x+I)\gamma a \subseteq J$ and $x\gamma a \in \underline{\rho_I}(J)$.
Again, let $p \in \underline{\rho_I}(J)$ and $a\gamma b \in M$. Then $p+I \subseteq J$. Now

$$(a\gamma(b+p)-a\gamma b)+I=(a\gamma(b+p)+I-a\gamma b \subseteq J$$

because J is an ideal of M. Hence $a\gamma(b+p)-a\gamma b \in \underline{\rho_I}(J)$. Thus $\underline{\rho_I}(J)$ is an ideal of M.

4 ROUGH NEAR-RINGS AND IDEALS

In this section we introduce the notion of rough Γ-near-rings and rough ideals and study some of their properties.

Definition 4.1. Let I be an ideal of M and $\rho_I(A)=(\underline{\rho_I}(A),\overline{\rho}_I(A))$ a rough set in the approximation space (M,I). If $\underline{\rho_I}(A)$ and $\overline{\rho}_I(A)$ are ideals (resp. sub Γ-near-rings) of M, then we call $\rho_I(A)$ *rough ideal (resp. Γ-near-ring).*

Note that a rough subΓ- near-ring is also called a rough Γ-near-ring. Clearly every rough ideal is a rough Γ-near-ring but the converse need not be true in general.

Lemma 4.2.

i) *Let I, J be two ideals of M, then $\rho_I(I)$ and $\rho_I(J)$ are rough ideals.*
ii) *Let I be an ideal and J a sub near-ring of M, then $\rho_I(J)$ is a rough near-ring.*

Proof. From Theorem 3.13 and Theorem 3.14, (i) and (ii) are clear.

Remark 4.3. If I is not an ideal and J is an ideal (resp. sub near-ring) of M, then $\rho_I(J)$ is not a rough ideal (resp. rough Γ-near-ring) which is shown in the following example.

Example 4.4. Let M= $\{0,a,b,c,x,y\}$ and $\Gamma=\{0,a,b,c,x\}$ Define addition and Γ in M as follows.

+	0	a	b	c	x	y
0	0	a	b	c	x	y
a	a	0	y	x	c	b
b	b	x	0	y	a	c
c	c	x	y	0	b	a
x	x	b	c	a	y	0
y	y	c	a	b	0	x

Γ	0	a	b	c	x
0	0	0	0	0	0
a	a	a	a	a	a
b	a	a	b	c	b
c	a	a	c	b	c
x	0	0	x	y	x

Then $(M,+,\Gamma)$ is a Γ- near-ring.

Let $I=\{a,c\}, J=\{0,x,y\}$. Clearly, J is an ideal and I is not an ideal(sub Γ-near-ring). Since $0+I=\{a,c\}, a+I=\{0,x\}, b+I=\{x,y\}, c+I=\{0,x\}, x+I=\{b,c\}$ and $y+I=\{a,c\}$, $\underline{\rho_I}(J)=\{a,b,c\}=\overline{\rho}_I(J)$. Thus both $\underline{\rho_I}(J)$ and $\overline{\rho}_I(J)$ are not ideals (sub near-rings) of M. Hence $\rho_I(J)$ is not a rough ideal (resp. rough Γ-near-ring).

Theorem 4.5. *Let I, J be two ideals of M and K be a sub Γ-near-ring of M. Then*

(i) $\overline{\rho}_I(K)\Gamma\,\overline{\rho}_J(K) \subseteq \overline{\rho}_{I+J}(K)$

(ii) $\underline{\rho}_I(K)\Gamma\underline{\rho}_J(K) = \underline{\rho}_{(I+J)}(K)$.

Proof. (i) Let $x \in \overline{\rho}_I(K)\Gamma\,\overline{\rho}_J(K)$. Then $x = p\gamma q$ for some $p \in \overline{\rho}_I(K)$ and $q \in \overline{\rho}_J(K)$. This means that there exist $y \in (p + I) \cap K$ and $z \in (q + J) \cap K$ and so $y\gamma z \in K$ and $xy \in (p + I).(q + J)$. This implies $y\gamma z \in (p\gamma q) + I + J$. Thus $(p\gamma q + I + J) \cap K \neq \emptyset$, and $x \in \overline{\rho}_{I+J}(K)$. Hence
$\overline{\rho}_I(K)\Gamma\,\overline{\rho}_J(K) \subseteq \overline{\rho}_{I+J}(K)$.
(ii) Let $p\gamma q \in \underline{\rho}_I(K)\Gamma\underline{\rho}_J(K)$. then p$\in \underline{\rho}_I(K)$ and $q \in \underline{\rho}_J(K)$ and so $(p + I) \subseteq K$ and $(q + J) \subseteq K$. Now $(p + I)(q + J) \subseteq K$ and $(p\gamma q + I + J \subseteq K$. This implies $p\gamma q \in \underline{\rho}_{(I+J)}(K)$.
On the other hand, since $I \subseteq I + J, J \subseteq I + J$, we have by Theorem 3.2(9). $\underline{\rho}_{(I+J)}(K) \subseteq \underline{\rho}_I(K)$ and

$$\underline{\rho}_{(I+J)}(K) \subseteq \underline{\rho}_J(K).$$
$$\underline{\rho}_{(I+J)}(K)\Gamma\underline{\rho}_{(I+J)}(K) \subseteq \underline{\rho}_I(K)\Gamma\underline{\rho}_J(K).$$

This means that $\underline{\rho}_{(I+J)}(K) \subseteq \underline{\rho}_I(K)\Gamma\underline{\rho}_J(K)$.

Theorem 4.4. Let I, J be two ideals of M and K a sub near-ring of M. Then

(i) $\underline{\rho}_{(I+J)}(K) = \underline{\rho}_I(K) + \underline{\rho}_J(K)$

(ii) $\overline{\rho}_{I+J}(K) = \overline{\rho}_I(K) + \overline{\rho}_J(K)$.

Proof. (i) Since $I \subseteq I + J$ and $J \subseteq I + J$, by Theorem 3.2(9) we have
$\underline{\rho}_{(I+J)}(K) \subseteq \underline{\rho}_I(K)$ and $\underline{\rho}_{(I+J)}(K) \subseteq \underline{\rho}_J(K)$.
Thus $\underline{\rho}_{(I+J)}(K) \subseteq \underline{\rho}_I(K) + \underline{\rho}_J(K)$.

Conversely assume that $k \in \underline{\rho}_I(K) + \underline{\rho}_J(K)$. Then $k = x + y$ for some $x \in \underline{\rho}_I(K)$ and $y \in \underline{\rho}_J(K)$. This means that $x + I \subseteq K$ and $y + J \subseteq K$. Consider

$$\begin{aligned} k + I + J &= x + y + I + J \\ &= x + y + I - y + y + J \\ &\subseteq x + I + y + J \\ &\subseteq K + K \subseteq K. \end{aligned}$$

Thus $k \in \underline{\rho}_{(I+J)}(K)$ and so $\underline{\rho}_I(K) + \underline{\rho}_J(K) \subseteq \underline{\rho}_{(I+J)}(K)$.
Thus $\underline{\rho}_I(K) + \underline{\rho}_J(K) = \underline{\rho}_{(I+J)}(K)$.
(ii) Since $I \subseteq I + J$ and $J \subseteq I + J$, by Theorem 3.2(9), we have
$\overline{\rho}_I(K) \subseteq \overline{\rho}_{I+J}(K)$ and $\overline{\rho}_J(K) \subseteq \overline{\rho}_{I+J}(K)$.
Therefore $\overline{\rho}_I(K) + \overline{\rho}_J(K) \subseteq \overline{\rho}_{I+J}(K)$.

Conversely assume that $y \in \overline{\rho}_{I+J}(K)$. Then $\big(y + (I + J)\big) \cap K \neq \emptyset$. Now there exists $j \in J$ such that

$$\begin{aligned} \big(y + (I + j)\big) \cap K &= (y + j - j + I + j) \cap K \\ &\subseteq (y + j + I) \cap K \neq \emptyset. \end{aligned}$$

This means that $y + j \in \overline{\rho}_I(K)$. Since $-j \in J$ and $(-j + J) \cap K = J \cap K \neq \emptyset$, being $0 \in J \cap K$, we have $-j \in \overline{\rho}_J(K)$. Consider $y = y + j - j \in \overline{\rho}_I(K) + \overline{\rho}_J(K)$. We have $\overline{\rho}_{I+J}(K) \subseteq \overline{\rho}_I(K) + \overline{\rho}_J(K)$. Thus $\overline{\rho}_I(K) + \overline{\rho}_J(K) = \overline{\rho}_{I+J}(K)$.

5.CONCLUSION

The theory of Γ-near ring and theory of rough sets have many application in various fields. In this paper is to present the concepts of congruence relation in Γ-near-rings and the lower and upper approximations of an ideal with respect to the congruence relation and introduce the notion of rough ideals in Γ-near-rings, which is a generalization of the notion of near rings. Some properties of the lower and upper approximations are discussed in Γ-near-rings. The definitions and results are extended to rings.

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