Points Certain Aspects of the Harmonic Functions with Respect to Symmetric

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Abstract- A wide variety of problems in engineering and physics involve harmonic functions, In particular harmonic functions provide optimal potential maps for robot navigation. In this paper, we obtain the coefficient conditions, extreme points, distortion bounds, and convolution for the subclass of harmonic functions with respect to symmetric points defined by Wright's generalized hyper geometric functions.

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1 Introduction

Harmonic functions are the solutions of the Laplaceâ€™s equation. Whereas these functions have been applied to scalar quantities such as temperature and conserved potentials in physical sciences, it is valid to extend the harmonic function concept and basic properties to vectors such as electrostatic field, magneto static field, current density, and gravitational force.

Harmonic functions offers a fast method of producing paths in a robot configuration space.The potential field method is relatively easy to implement, has good reactivity in dynamic environment. However, substantial problems with the method, specially in the case of complex environments.

The use of harmonic functions for the potential solves this problem. Harmonic functions satisfy the min-max principle and hence unplanned creation of local mimima within the solution region is impossible.

Let $\mathcal{A}$ denote the family of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$ is of the form,

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1.1)$$

and satisfies the normalization condition $f(0) = f'(0) - 1 = 0$.

Let $\mathcal{H}$ be the family of all harmonic functions of the form

$$= h + \overline{g} \quad (1.2)$$

where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1, \quad (z \in \mathbb{U}) \qquad (1.3)$$

are in the class $\mathcal{A}$. We call h the analytic part and g the co-analytic part of f. Clunie and Sheil-Small [Clunie Sheil-Small 1984] investigated the class $\mathcal{H}$ as well as its geometric subclasses and obtained some coefficient bounds. A necessary and sufficient condition for f to be locally univalent and sense preserving in $\mathcal{H}$ is that $|h'(z)| > |g'(z)|$ in $\mathcal{H}$ [Clunie Sheil-Small 1984].

Hence

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n z^n}, \quad |z| < 1. \qquad (1.4)$$

Let $\overline{\mathcal{H}}$ be the subclass of $\mathcal{H}$ consisting of functions $f = h + \overline{g}$ so that the functions h and g take the form

$$f(z) = z - \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n z^n}, \quad |z| < 1. \qquad (1.5)$$

Let the Hadamard product (or convolution) of two power series

$$\phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n$$

and

$$\Psi(z) = z + \sum_{n=2}^{\infty} \Psi_n z^n$$

is defined by

$$(\phi * \Psi)(z) = z + \sum_{n=2}^{\infty} \phi_n \Psi_n z^n = (\Psi * \phi)(z)$$

Let $\alpha_1, A_1, \ldots, \alpha_q, A_q$ and $\beta_1, B_1, \ldots, \beta_s, B_s$ $(q, s \in \mathbb{N})$ be positive and real parameters such that

$$1+\sum_{j=1}^{s}\ {}_{j}-\sum_{j=1}^{q} A_{j} \geq 0.$$

The Srivastava-Wright generalized hypergeometric function [Wright 1940] is defined by,

$${}_q\Psi[(\alpha_i, A_i)_q; (\beta_i, B_i)_s; z] :=$$

$${}_q\Psi[(\alpha_1, A_1), \dots, (\alpha_q, A_q); (\beta_1, B_1), \dots, (\beta_s, B_s); z] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{q} \Gamma(\alpha_i+nA_i)}{\prod_{i=1}^{s} \Gamma(\beta_i+nB_i)} \frac{z^n}{n!} \quad (1.6)$$

for $z \in \mathbb{U}$. If $A_i = 1 (i = 1 \dots, q)$ and $B_i = 1 (i = 1, \dots, s)$, we have the relationship,

$$\Omega\, {}_q\Psi[(\alpha_i, 1)_q; (\beta_i, 1)_s; z] = {}_qF_s(\alpha_1 \dots \alpha_q; \beta_1 \dots \beta_s; z)$$

where ${}_qF_s(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$ is the generalized hypergeometric function [Srivastava Karlsson 1985] and

$$\Omega = \Omega(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s) := \frac{\prod_{i=1}^{s} \Gamma(\beta_i)}{\prod_{i=1}^{q} \Gamma(\alpha_i)}. \quad (1.7)$$

Dziok and Srivastava [Dziok Srivastava 2003] introduced the linear operator by using the generalized hypergeometric function and subsequently Dziok and Raina [Dziok Raina 2004] extended the linear operator by using Wright generalized hypergeometric function.

Let

$$\mathcal{W}_{q,s}[(\alpha_i, A_i)_q; (\beta_i, B_i)_s]: S_H \to S_H,$$

be a linear operator defined by

$$\mathcal{W}_{q,s}[(\alpha_i, A_i)_q; (\beta_i, B_i)_s]f(z) = \{\Omega z\, {}_q\Psi_s[(\alpha_i, A_i)_q; (\beta_i, B_i)_s; z]\} * f(z)$$

We observe that, for a function f(z) of the form (1.2), we have

$$\mathcal{W}_{q,}[(\alpha_i, A_i); (, B_i)_s]f(z) = z + \sum_{n=2}^{\infty} \Omega\sigma_n(\alpha_1)a_n z^n \quad (1.8)$$

where Ω is given by (1.7) and $\sigma_n\alpha_1$ is defined by

$$\sigma_n(\alpha_1) = \frac{\Gamma(\alpha_1+A_1(n-1))\ldots\Gamma(\alpha_q+A_q(n-1))}{(n-1)!\Gamma(\beta_1+B_1(n-1))\ldots\Gamma(\beta_s+B_s(n-1))} \quad (1.9)$$

For convenience, we use the slender notation $\mathcal{W}_{q,}$ to represent the following:

$$\mathcal{W}_{q,s}[\alpha_1, A_1, B_1]f(z) = \mathcal{W}_{q,s}[(\alpha_1, A_1), \ldots, (\alpha_q, A_q); (\beta_1, B_1), \ldots, (\beta_s, B_s)]f(z)$$

The Wright generalized hypergeometric function contains, Dziok-Srivastava operator as its special cases, further other linear operators the Hohlov operator, the Carlson-Shaffer operator [Carlson Shaffer 1984], the Ruscheweyh derivative operator [Ruscheweyh 1975], the generalized Bernardi-Libera-Livingston operator, the fractional derivative operator [Srivastava Owa 1987], and so on.

Applying the Srivastava-Wright operator to the harmonic functions $f = h + \bar{g}$ given by (1.2) we obtain

$$\mathcal{W}_{q,}[\alpha_1, A_1, B_1](z) = \mathcal{W}_{q,s}[\alpha_1, A_1, B_1]h(z) + \mathcal{W}_{q,s}[\alpha_1, A_1, B_1]\overline{g(z)} \quad (1.10)$$

and we call this as Wright generalized operator on harmonic function.

We introduce a new subclass $\mathcal{W}_{q,}([\alpha_1, A_1, B_1], \gamma)$ of $\mathcal{H}$ which is defined as following,

Definition 1 *For $0 \le \gamma < 1$ and $z = re^{i\theta}$, let $f \in \mathcal{H}$, then $f \in \mathcal{HS}_s^*([\alpha_1, A_1, B_1], \gamma)$ is said to be harmonic starlike with respect to symmetric points, which satisfying the condition*

$$\Re\left\{(1 + e^{i\alpha})\frac{2(\mathcal{W}_{q,}[\alpha_1,A_1,B_1]f(z))'}{\mathcal{W}_{q,s}[\alpha_1,A_1,B_1]f(z) - \mathcal{W}_{q,s}[\alpha_1,A_1,B_1]f(-z)} - e^{i\alpha}\right\} > \gamma \quad (1.11)$$

where $f'(z) = \frac{\partial}{\partial\theta}(z = re^{i\theta})$.

Also let, $\mathcal{HS}_s^*(\alpha_1, A_1, B_1) = \mathcal{HS}_s^*(\alpha_1, A_1, B_1) \cap \mathcal{H}$.

Definition 2 *For harmonic functions of the form:*

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \bar{z}^n, \quad |b_1| < 1 \qquad (1.12)$$

and

$$G(z) = z - \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} B_n \bar{z}^n, \quad (A_n \geq 0; \quad B_n \geq 0) \qquad (1.13)$$

We define the convolution of two harmonic functions as

$$(f * G)(z) = f(z) * G(z) = z - \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} b_n B_n \bar{z}^n \qquad (1.14)$$

2 Coefficient characterization

We begin with a sufficient condition for functions in $\mathcal{HS}^*([\alpha, A, B], \gamma)$.

Theorem 1 *Let* $f = h + g$ *be given by* (1.4). *If*

$$\sum_{n=2}^{\infty} \frac{4n-(1+\gamma)[1-(-1)^n]}{2(1-\gamma)} \Omega\sigma_n(\alpha_1)|a_n| + \sum_{n=1}^{\infty} \frac{4n+(1+\gamma)[1-(-1)^n]}{2(1-\gamma)} \Omega\sigma_n(\alpha_1)|b_n| \le 1, \quad (2.1)$$

Where a_1=1 and $0 \le \gamma < 1$,then f is sense preserving, harmonic univalent in $\mathbb{U}$ and $f \in \mathcal{HS}^*([\alpha_1, A_1, B_1], \gamma)$.

Proof. According the condition (1.11), we only need to show that if (2.1) holds,then

$$\Re\left\{(1+e^{i\alpha})\frac{2(\mathcal{W}_{q,s}[\alpha_1,A_1,B_1]f(z))}{\mathcal{W}_{q,s}[\alpha_1,A_1,B_1]f(z)-\mathcal{W}_{q,s}[\alpha_1,A_1,B_1]f(-z)} - e^{i\alpha}\right\} = \Re\left(\frac{A(z)}{B(z)}\right) > \gamma$$

where $A(z) = (1+e^{i\alpha})2z(\mathcal{W}_{q,s}[\alpha_1, A_1, B_1]f(z))$
$-e^{i\alpha}((\mathcal{W}_{q,s}[\alpha_1, A_1, B_1]f(z))f(z) - (\mathcal{W}_{q,s}[\alpha_1, A_1, B_1]f(z))f(-z))$

Or equivalently,

$$= 2z + \sum_{n=2}^{\infty}[2n + (2n - (1-(-1)))]\Omega\sigma_n(\alpha_1)a_n z^n$$
$$- \sum_{n=1}^{\infty}[2n + (2n + (1-(-1)))]\Omega\sigma_n(\alpha_1)b_n z^n.$$

and $n=1$ n 1 n

$(z) = W_q, [\alpha_1, A_1, B_1](z) - W_{q,s} [\alpha_1, A_1, B_1]f(-z)$

$= \; 2z + \; (1-(-1))\Omega\sigma\,() + \sum^\infty \; (1-(-1))\Omega\sigma\,()$

$\sum^\infty$

$n=2$ n 1 n $n=1$ n 1 n

Using the fact that $\Re(w) > \gamma$ if and only if $|1-\gamma+w| \geq |1+\gamma-w|$, it is sufficient to show that

$$|(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)| \geq 0.$$

(2.2

) substituting for A(z) and B(z) in (2.2), we get

$|(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)|$

$= \; |2(2-\gamma)z + \sum^\infty \; [4n - (1-(-1)^n) + (1-\gamma)(1-(-1)^n)]\Omega\sigma\,(\alpha\,)a\; z^n$

$n=2$ n 1 n

$-\sum^\infty \; [4n + (1-(-1)) - (1-\gamma)(1-(-1)^n)]\Omega\sigma\,(\alpha\,)b\; z^n|$

$n=1$ n 1 n

$-|-2\gamma z + \sum^\infty \; [4n - (1-(-1)) - (1+\gamma)(1-(-1)^n)]\Omega\sigma\,(\alpha \quad z^n$

$)a$

$n=2$ n 1 n

$-\sum^\infty \; [4n + (1-(-1)) + (1+\gamma)(1-(-1)^n)]\Omega\sigma\,(\alpha\,)b\; z^n|$

$n=1$ n 1 n

$\geq \; 4(1-\gamma)|z| - 2\sum^\infty \; [4n - (1+\gamma)(1-(-1)^n)]\Omega\sigma\,(\alpha \quad ||z|^n$

$)|a$

$n=2$ n 1 n

$-2\sum^\infty \; [4n + (1+\gamma)(1-(-1)^n)]\Omega\sigma\,(\alpha\,)|b\; ||z|^n$

$n=1$ n 1 n

$\geq \; 4(1-\gamma)|z|\,(1 - \; \dfrac{[4n-(1+\gamma)(1-(-1)^n)]}{2(1-\gamma)}\; \Omega\sigma\,()|a\,|$

$\sum^\infty$

$n=2$ n 1 n

$-\sum^{\infty}$ $[4n+(1+\gamma)(1-(-1)^{n})]$ $\Omega\sigma$ $()|b\ |)$

$n=1$ $2(1-\gamma)$ $n\ 1\ n$

$\geq\ 0$

by virtue of the inequality (2.2), This implies that $f \in \mathcal{HS}^*([\alpha, A, B], \gamma)$.

$s\ \ 1\ 1\ \ 1$

Theorem 2 *For* $a_1=1$, $0 \leq \gamma < 1$, *and let* $f = h + g$ *be given by* (1.5) *then*

$f \in \overline{\mathcal{HS}}^*([\alpha, A, B], \gamma)$ if and only if

$s\ \ 1\ 1\ 1$

$\sum^{\infty}$ $4n-(1+\gamma)(1-(-1)^{n})$ $\Omega\sigma$ $(\alpha\ )|a$ $|+$ $\sum^{\infty}$ $4n+(1+\gamma)(1-(-1)^{n})$ $\Omega\sigma$ $(\alpha\ )|b\ | \leq 1$ (2.3)

$n=2$ $2(1-\gamma)$ $n\ 1\ n$ $n=1$ $2(1-\gamma)$ $n\ 1\ n$

Proof. Since $\mathcal{HS}^*([\alpha, A, B], \gamma) \subset \mathcal{HS}^*([\alpha, A, B], \gamma)$, we only need to prove the "only if " part of the

$s\ \ 1\ 1\ \ 1$ $s\ \ 1\ 1\ 1$

theorem. To this end, for functions (z) of the form (1.5), we notice that the condition

$\Re\left((1+e^{i\alpha})\frac{2zW_{q,}([\alpha_1,A_1,B_1]f(z))}{W_{q,s}([\alpha 1,A1,B1]f(z))-W_{q,s}([\alpha 1,A1,B1]f(-z))} - e^{i\alpha}\right) > \gamma$ (2.4)

$= \Re\ \{^{1}\ (2(1-\gamma) - \sum^{\infty}$ $(4n-(1+\gamma)(1-(-1)^{n}))\Omega\sigma\ ()|a\ \ |z^{n}$

$-$

Y $n=$ n 1 n

$-\sum^{\infty}$ 2

$(4n + (1+\gamma)(1-(-1)^n))\Omega\sigma\,()|b\ |z^n)\}$ –

$n=1$ n 1 n

Where Y = $2z -$ $(1-(-1))\Omega(\alpha\ )|a\ \ |z^n + \sum^{\infty}$ $(1-(-1)^n)\Omega\sigma\,(\alpha\ )|b\ |z^n$.
$\sum^{\infty}$

$n=2$ n 1 n $n=1$ n 1 n –

The above condition must hold for all values of z, $|z| = r < 1$. Upon choosing the values of z on the positive real axis where $0 \le z = r < 1$, the above inequality reduces to

1 $(2(1-\gamma) -$ $(4n - (1+\gamma)(1-(-1)^n))\Omega\sigma\,()|a\ \ |r^{n-1}$
$\sum^{\infty}$

—

Y_1 n 1 n (2.5)

$n=$

2

$-\sum^{\infty}$ $(4n + (1+\gamma)(1-(-1)^n))\Omega\sigma\,(\alpha\ )|b\ |r^{n-1})$

$n=1$ n 1 n

where $Y_1 = 2z -$ $_{n=2}$ $(1-(-1))\Omega(\alpha\ )|a\ \ |r^{n-1} + \sum^{\infty}$ $(1-(-1)^n)\Omega\sigma\,(\alpha\ )|b\ |r^{n-1}$.
$\sum^{\infty}$

If condition (2.3) does not hold then the numerator in (2.5) is negative for r sufficiently close to 1. Thus there exists $z_0 = r_0$ in (0,1) for which the quotient in (2.5) is negative.This contradicts the required condition for $\overline{\mathcal{HS}}^*(\alpha, \gamma)$ and so the proof is completed.

3 Extreme points and Distortion theorem

Next, we determine the extreme points of closed hulls of $\overline{\mathcal{HS}}^*_s([\alpha_1, A_1, B_1], \gamma)$ denoted by $clco\ \overline{\mathcal{HS}}^*_s([\alpha_1, A_1, B_1], \gamma)$.

Theorem 3 *A function* $f \in clco\ \overline{\mathcal{HS}}^*_s([\alpha_1, A_1, B_1], \gamma)$ if and only if $f(z)$ can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} (X_n h_n + Y_n g_n) \tag{3.1}$$

where $h_1(z) = z$,

$$h_n(z) = z - \frac{2(1-\gamma)}{4n-(1+\gamma)(1-(-1)^n)} z^n \quad (n \geq 2),$$

and

$$g_n(z) = z + \frac{2(1-\gamma)}{4n+(1+\gamma)(1-(-1)^n)} z^n \quad (n \geq 1),$$

$X_n \geq 0$, $Y_n \geq 0$, $\sum_{n=1}^{\infty}$

$(X_n + Y_n) = 1$

In particular, the extreme points of $\mathcal{HS}^*([\alpha_1, A_1, B_1], \gamma)$ are $\{h_n\}$ and $\{g_n\}$.

Proof. For functions $f(z)$ having the form (3.1), we have

$$f(z) = \sum_{n=1}^{\infty} (X_n h_n + Y_n g_n)$$

$$= \sum_{n=1}^{\infty} (X_n h_n + Y_n g_n) z - \sum_{n=2}^{\infty} \frac{2(1-\gamma)}{4n-(1+\gamma)(1-(-1)^n)} X_n z^n$$

$$+ \sum_{n=1}^{\infty} \frac{2(1-\gamma)}{4n+(1+\gamma)(1-(-1)^n)} X_n z^n$$

Then by Theorem2,

$$\sum_{n=2}^{\infty} \frac{4n-(1+\gamma)(1-(-1)^n)}{2(1-\gamma)} \Omega_\sigma(\alpha_1)|a_n| + \sum_{n=1}^{\infty} \frac{4n+(1+\gamma)(1-(-1)^n)}{2(1-\gamma)} \Omega_\sigma(\alpha_1)|b_n|$$

$$= \sum_{n=2}^{\infty} \frac{4n-(1+\gamma)(1-(-1)^n)}{2(1-\gamma)} \Omega_\sigma(\alpha_1) \frac{2(1-\gamma)}{4n-(1+\gamma)(1-(-1)^n)} X_n$$

$$+\sum_{n=1}^{\infty}\frac{4n+(1+\gamma)(1-(-1)^n)}{2(1-\gamma)}\Omega\sigma_n(\alpha_1)\frac{2(1-\gamma)}{4n+(1+\gamma)(1-(-1)^n)}Y_n$$

$$=\sum_{n=2}^{\infty}X_n+\sum_{n=1}^{\infty}Y_n$$

$$=1-X_1$$

$$\leq 1$$

Therefore $f\in clco\,\mathcal{HS}^*([\alpha_1,A,B],\gamma)$. on the converse, we suppose $f\in \overline{clco}\,\mathcal{HS}^*([\alpha_1,A,B],\gamma)$.

By setting

$$X_n=\sum_{n=2}^{\infty}\frac{4n-(1+\gamma)(1-(-1)^n)}{2(1-\gamma)}\Omega\sigma_n(\alpha_1)|a_n|,\ (n\geq 2),$$

and

$$Y_n=\sum_{n=1}^{\infty}\frac{4n+(1+\gamma)(1-(-1)^n)}{2(1-\gamma)}\Omega\sigma_n(\alpha_1)|b_n|,\ (n\geq 1).$$

From Theorem 2, we can deduce that $0\leq X_n\leq 1$, $(n\geq 2)$ and $0\leq Y_n\leq 1$, $(n\geq 1)$. We define $X_1=1-\sum_{n=2}^{\infty}X_n-\sum_{n=1}^{\infty}Y_n$. Again from Theorem 2, $X_1\geq 0$. Therefore, $f(z)=\sum_{n=1}^{\infty}(X_n+Y_n)$ as required.

Theorem 4 *Let the functions* (z) *defined by* (1.5) *be in the class* $\mathscr{HS}^*([\alpha, A, B], \gamma)$. *Then for* $|z| =$ s 1 1 1

$r < 1$, *we*

have

$$(1 - |b\ |)r - \quad 1 \quad (^{1-\gamma} - {}^{2+(1+\gamma)} \, |b\ |) \, r^2 \leq |f(z)|$$

1 $\Omega\sigma 2(\alpha 1)$ 4 4 1

$$\leq (1 + |\ \ |)r + \text{———} (^{1-\gamma} - {}^{2+(1+\gamma)} \, |b\ |) \, r^2$$

1

1 $\Omega\sigma 2(\alpha 1)$ 4 4 1

Proof. We only prove the right-hand side inequality.The proof for the left-hand inequality is similar, and will be omitted.

Let $f \in \mathscr{HS}^*([\alpha, A, B], \gamma)$. Taking the absolute value of (z) we have

s 1 1 1

$|(z)| \leq (1 + |b_1|)r + \sum_{n=2}^{\infty} (|a \quad | + |b_n|)r^n$

$\leq (1 + |b_1|) + \sum_{n=2}^{\infty} \quad n \quad | + |b_n|)^2$

$(|a$ $\quad \Omega(\alpha 1)$ $(|a$

$= (1 + |b \,|) + {}^{(1-\gamma)}$

n

$\sum$

∞ $\quad | + |b$

$1 \quad \Omega\sigma 2(\alpha 1) \quad n \quad (1-\gamma) \quad n \quad n$

$=2$

$\leq (1 + |b \,|) + {}^{(1-\gamma)2} \quad ({}^{4n-(1+\gamma)(1-(-1)^n)} \, |a \,|$

$\sum^{\infty}$

$1 \quad \Omega\sigma 2(\alpha 1) \quad n \quad 4(1-\gamma) \quad n$

$=2$

$+ \quad |) \; \Omega\sigma \; ()$

$4n+(1+\gamma)(1-(-1)$

${}^{n)} \; |b$

$4(1-\gamma) \quad n \quad k \quad 1$

$\leq (1 + |b \,|) + {}^{(1-\gamma)2} \sum^{\infty} \quad ({}^{4n-(1+\gamma)(1-(-1)^n)} \, |a \,|$

$1 \quad 2\Omega\sigma 2(\alpha 1) \quad n \quad 2(1-\gamma) \quad n$

$=2$

$+ \quad) \; |b$

$4n+(1+\gamma)(1-(-1)^n$

|) Ωσ ()

2(1−γ) n k 1

≤ (1 + |b |) + (1−γ)2 (1 − 3(1+γ) |b |)

1 2Ωσ2(α1) 1−γ 1

= (1 + |b |) + (1−γ − 3(1+γ) |b |) r2

1

1 Ωσ2(α1) 2 2 1

4 Convolution and Convex combination

Theorem 5 *For* 0 ≤ μ ≤ γ < 1, *let* f ∈ 𝒽𝒮*([α , A , B], γ) *and* G ∈ 𝒽𝒮*([α , A , B], γ). *Then for f* ∗

s 1 1 1 s 1 1 1

G ∈ 𝒽𝒮*([α , A , B], γ) ⊂ 𝒽𝒮*([α , A , B], μ).

s 1 1 1 s 1 1 1

Proof. Let the function (z) defined by (1.12) be in the class 𝒽𝒮*([α , A , B], γ) and let the function

s 1 1 1

(z) defined by (1.13) be in the class 𝒽𝒮*([α , A , B], μ).We wish to show that the coefficients of f ∗ G satisfy

s 1 1 1

the required condition given in Theorem 2.For G ∈ 𝒽𝒮*([α , A , B], μ), we note that 0 ≤ A ≤ 1 and 0 ≤ B ≤

1. Now the convolution function f ∗ G we obtain,

s 1 1 1 n n

$$\sum_{n=2}^{\infty} 4n-(1+\gamma)(1-(-1)^{n})\Omega\sigma(\alpha_{n\ 1})|a_{n}||A_{n}|$$

$$+\sum_{n=1}^{\infty} 4n+(1+\gamma)(1-(-1)^{n})\Omega\sigma(\alpha_{n\ 1})|b_{n}||B_{n}|$$

$$\leq \sum_{n=2}^{\infty} 4n-(1+\gamma)(1-(-1)^{n})\Omega\sigma(\alpha_{n\ 1})|a_{n}|$$

$$+\sum^{\infty} 4n+(1+\gamma)(1-(-1)^{n})\Omega\sigma(\alpha\ )|b\ |$$

$n=1$

$\leq 2(1-\gamma)$

Since $0 \leq \mu \leq \gamma < 1$ and $f \in \mathcal{HS}^*([\alpha_1, A_1, B_1], \gamma)$.

Therefore $f * G \in \mathcal{HS}^*_s([\alpha_1, A_1, B_1], \gamma) \subset \mathcal{HS}^*_s([\alpha_1, A_1, B_1], \mu)$,

since the above inequality bounded by $2(1-\gamma)$ while $2(1-\gamma) \leq 2(1-\mu)$.

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