The Primary Elements of the Laws of Motion – An Analysis

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Abstract

This paper deals with studying and analyzing primary elements of Newton's laws of motion and its interpretations. Sir Isaac Newton's three laws of motion describe the motion of massive bodies and how they interact. While Newton's laws may seem obvious to us today, more than three centuries ago they were considered revolutionary. Newton was one of the most influential scientists of all time. His ideas became the basis for modern physics. He built upon ideas put forth from the works of previous scientists including Galileo and Aristotle and was able to prove some ideas that had only been theories in the past. He studied optics, astronomy and math — he invented calculus. (German mathematician Gottfried Leibniz is also credited with developing it independently at about the same time.) Newton is perhaps best known for his work in studying gravity and the motion of planets. Urged on by astronomer Edmond Halley after admitting he had lost his proof of elliptical orbits a few years prior, Newton published his laws in 1687, in his seminal work "Philosophiæ Naturalis Principia Mathematica" (Mathematical Principles of Natural Philosophy) in which he formalized the description of how massive bodies move under the influence of external forces. In formulating his three laws, Newton simplified his treatment of massive bodies by considering them to be mathematical points with no size or rotation. This allowed him to ignore factors such as friction, air resistance, temperature, material properties, etc., and concentrate on phenomena that can be described solely in terms of mass, length and time. Consequently, the three laws cannot be used to describe precisely the behavior of large rigid or deformable objects; however, in many cases they provide suitably accurate approximations.

Newton's laws pertain to the motion of massive bodies in an inertial reference frame, sometimes called a Newtonian reference frame, although Newton himself never described such a reference frame. An inertial reference frame can be described as a 3-dimensional coordinate system that is either stationary or in uniform linear motion., i.e., it is not accelerating or rotating. He found that motion within such an inertial reference frame could be described by three simple laws. In particular, laws and qualities must be intelligible in terms of the shape, size, motion and impenetrability (or solidity) of bodies. In this way, one might conclude that Locke and Leibniz actually do not necessarily disagree on whether gravity can be made intelligible in mechanist terms; they simply disagree on the propriety of the contention that God could "superadd" a feature to bodies that cannot be made intelligible in that way.

Key words: Newton Locke and Leibniz, Principia, motion of massive bodies

Introduction

Locke's point of view, we know that human beings—which are, or at least contain, material bodies with size, shape, motion and solidity, along with parts characterized by those qualities—are capable of thought, but since we cannot discern how any material thing could possibly have that capacity, we conclude that God may have superadded that feature to us, or to our bodies. Thought and gravity are dis-analogous in the sense that we did not require anything like Newton's theory to convince us that human beings can think, but they are otherwise analogous. Newton then attempts to make the following argument: since Leibniz would have to agree that thinking is not a mechanical process, and not mechanically explicable, he must agree that there is at least one aspect of the world that has the following two features, (1) it is not mechanical; and, (2) it is clearly not to be rejected on that ground alone. He attempts to liken *gravity* (as he understands it) to *thinking* (as he believes Leibniz is required to understand it), arguing that despite the fact that it is not mechanical—it cannot be explained mechanically—it should not be rejected on that ground. This argument may be predicated on the view that human beings, material things, or at least partially material things, do the thinking, rather than immaterial things, such as minds or souls, for if one attributes all thought to an immaterial mind or soul, then there is no pressure to say that anything in nature, or perhaps even any aspect of anything in nature, has a feature that cannot be mechanically explicated. If one accepts Locke's view (apparently also endorsed by Newton) that we should attribute thinking to material things, or to aspects of material things, then perhaps Newton has successfully followed Locke in likening gravity to thought, thereby making room for aspects of nature that are not mechanical after all. This vexing issue would continue to generate debates amongst Newton's and Leibniz's various followers in England, and on the Continent, respectively.

Leibniz's most extensive debate with the Newtonians would not occur until the very end of his life. His celebrated correspondence with Samuel Clarke, Newton's friend and supporter in London in the early part of the eighteenth century, is his most famous interaction with the Newtonians, occurring right before his death in 1716 (Clarke and Leibniz 1717). Leibniz fomented the correspondence in November of 1715 by sending a short, provocative letter to Princess Caroline of Wales, one designed to provoke a response from Newton's circle in London. Leibniz knew well that Princess Caroline was a leading intellectual and political figure in England at the time, one who would surely wish to see the views of her countrymen defended against Leibniz's rather shocking claims about the religious consequences of Newtonian thinking. He opens his initial letter by mentioning both Locke and Newton, along with the issues about materiality and thinking that arose in his near exchange with Newton in 1712:

> *Natural religion itself seems to decay [in England] very much. Many will have human souls to be material; others make God himself a corporeal being. Mr. Locke and his followers are uncertain at least whether the soul is not material and naturally perishable. Sir Isaac Newton says that space is an organ which God makes use of to perceive things by. But if God stands in need of any organ to perceive things by, it will follow that they do not depend altogether on him, nor were produced by him. (Clarke and Leibniz 1717: L 1: 1–3)*

Objective:

This paper intends to explore three physical laws that, together, laid the foundation for classical mechanics. They describe the relationship between a body and the forces acting upon it, and its motion in response to those forces.

Newton ideas and methods

In many ways, Newton eventually succeeded in convincing that his own ideas and methods were superior to those of the Cartesians—especially when it came to thinking about motion and its causes within nature—but this historical fact did not mean that Newtonianism, even broadly construed, became the dominant trend in natural philosophy during Newton's lifetime. On the contrary, Newton's views continued to be the subject of intense scrutiny and debate, especially amongst Leibniz and his followers (such as Christiaan Wolff) and fellow mechanists (such as Huygens). Indeed, a late-seventeenth-century debate between Cartesian and Newtonian ideas was supplanted by an early eighteenth century debate between Leibnizian and Newtonian views; the latter debate would continue in one form or another for the rest of the century: it was a driving force during the French Enlightenment, and remained a powerful stimulant to philosophical theorizing in the 1770s and 1780s, when Kant forged his magisterial "critical" system of philosophy, an approach that almost single-handedly set the philosophical agenda of the early nineteenth century. Hence Newton's influence on the eighteenth century did not take the form of a single philosophical program or movement; instead, it was the controversial nature of his ideas and methodology that drove much of the philosophical discussion.

There are some irreducibly nationalist elements in the way that philosophy developed over the course of the eighteenth century, so it may be reasonable to chart Newton's impact country by country. Newton's ideas and methods were certainly most influential in England, where there grew to be a strong "Newtonian" movement—also called the "experimental philosophy" program—by roughly 1700. By the *fin de siecle*, it is probably safe to say that natural philosophy had become heavily Newtonian in England, at least in the sense that it had eclipsed both Cartesianism (Henry 2013: 124 and introduction to Voltaire 1738/1992: 7), and other local movements, such as Cambridge Platonism, which had exhibited a strong influence during the previous generation. One might put the point somewhat differently: to the extent that there was a dominant strand in England by 1700, it was the "experimental philosophy", a view that was associated strongly with figures such as Boyle, Newton and Locke. Figures such as Hobbes had opposed this approach to solving philosophical problems, but had failed to gain nearly as much influence.

The First Law of Motion states, "A body at rest will remain at rest, and a body in motion will remain in motion unless it is acted upon by an external force." This simply means that things cannot start, stop, or change direction all by themselves. It takes some force acting on them from the outside to cause such a change. This property of massive bodies to resist changes in their state of motion is sometimes called inertia.

The Second Law of Motion describes what happens to a massive body when it is acted upon by an external force. It states, "The force acting on an object is equal to the mass of that object times its acceleration." This is written in mathematical form as F = ma, where F is force, m is mass, and a is acceleration. The bold letters indicate that force and acceleration are vector quantities, which means they have both magnitude and direction. The force can be a single force, or it can be the vector sum of more than one force, which is the net force after all the forces are combined. When a constant force acts on a massive body, it causes it to accelerate, i.e., to change its velocity, at a constant rate. In the simplest case, a force applied to an object at rest causes it to accelerate in the direction of the force. However, if the object is already in motion, or if this situation is viewed from a moving reference frame, that body might appear to speed up, slow down, or change direction depending on the direction of the force and the directions that the object and reference frame are moving relative to each other.

The Third Law of Motion states, "For every action, there is an equal and opposite reaction." This law describes what happens to a body when it exerts a force on another body. Forces always occur in pairs, so when one body pushes against another, the second body pushes back just as hard. For example, when you push a cart, the cart pushes back against you; when you pull on a rope, the rope pulls back against you; when gravity pulls you down against the ground, the ground pushes up against your feet; and when a rocket ignites its fuel behind it, the expanding exhaust gas pushes on the rocket causing it to accelerate. If one object is much, much more massive than the other, particularly in the case of the first object being anchored to the Earth, virtually all of the acceleration is imparted to the second object, and the acceleration of the first object can be safely ignored.

For instance, if you were to throw a baseball to the west, you would not have to consider that you actually caused the rotation of the Earth to speed up ever so slightly while the ball was in the air. However, if you were standing on roller skates, and you threw a bowling ball forward, you would start moving backward at a noticeable speed. The three laws have been verified by countless experiments over the past three centuries, and they are still being widely used to this day to describe the kinds of objects and speeds that we encounter in everyday life. They form the foundation of what is now known as classical mechanics, which is the study of massive objects that are larger than the very small scales addressed by quantum mechanics and that are moving slower than the very high speeds addressed by relativistic mechanics.

Definition of Force

When a constant force acts on a body, the forces result in the acceleration of the body. However, if the object is already in motion, or if this situation is viewed from a moving frame of reference, the body might appear to speed up or slow down or change its direction depending on the direction of the force.

Mathematically, we express the law as follows:

$f \propto dPdt \Rightarrow f \propto mv-mut \Rightarrow f \propto m(v-u)t \Rightarrow f \propto ma \Rightarrow f=kma$

Where k is the constant of proportionality and it comes out to be 1 when the values are taken in SI unit. Hence the final expression will be,

$F=ma$

Perhaps more importantly, Newton's view of motion, his understanding of space and time, and his approach to achieving knowledge of natural phenomena, helped to shape the agenda of British philosophy for the next fifty years. In addition to Newton's influence on Locke's thinking about matter and causation, explored above, both Berkeley and Hume expended considerable energy grappling with the wider consequences and implications of the Newtonian version of the experimental philosophy. For his part, Berkeley famously derided many Newtonians methods and ideas—sometimes exempting Newton himself from his conception of the worst philosophical excesses of his followers—including the rise of the calculus among mathematicians (in *The Analyst*) and the use of the idea of a force as the basic causal concept in natural philosophy (in *De Motu*—both reprinted in Berkeley 1992). Berkeley's theory of ideas, which arose in part from his reflections of what we would now call Locke's “empiricist” notion of representation, suggested to him that no idea can be abstract: each idea must represent a *particular* rather than a universal. Hence we can have an idea of a particular car, but not of a car in general (not of, as it were, the form of a car); we can have an idea of a particular shade of yellow, perhaps because we've just seen a lovely yellow rose at the florist, but not of yellow in general; and so on. Berkeley then argued that modern mathematics, especially the calculus, and modern natural philosophy, especially Newtonian versions of it, were often reliant on abstract ideas, and therefore philosophically suspect. For instance, he contended that the very idea of absolute motion was suspect because we can represent to ourselves only various motions with particular features related to particular bodies in motion, but “absolute” motion cannot be rendered *particular* in anything like this way; it remains abstract (Downing 2005: 235). Thus although Newtonian views were considered to be essential to the rise of experimental philosophy in Britain, Berkeley derided them as insufficiently experimental, as overly reliant on representations of universals and of universal quantities, rather than on the representation of particulars. In a reflection of Malebranche's influence, Berkeley also argued that some Newtonians wrongly attributed genuine causal powers to ordinary material objects through their use of the concept of impressed force; wrongly, because Berkeley firmly rejected the notion that any body could exert any causal power. All causation in Berkeley's system is due either to the intervention of the divine in the course of history, or to spirits or minds, which are genuinely causally active. Finally, in an argument that would prefigure Mach's reactions to Newtonian conceptions of space, time and motion in the late nineteenth century—which were expressive of a broad commitment to “empiricism”—Berkeley contended that absolute space is a metaphysical aberration: philosophers should not posit any entity or thing that is beyond all possible perception. In sum, Berkeley was highly critical of many aspects of the Newtonian program, but for that very reason, it was Newton's ideas that helped to shape many of his philosophical projects.

Motion and relativity

For his part, Hume had a more nuanced reaction to the emergence of the Newtonian program (cf. Schliesser 2007 and DePierris 2012). He certainly signaled his endorsement of the experimental philosophy—itself strongly associated with the Newtonians, along with figures like Boyle and Hooke, as we have seen—when he gave his *Treatise* the following subtitle: “being an attempt to introduce the experimental method of reasoning into moral subjects”. And one might argue that Hume made a kind of Lockean move when he chose to endorse the Newtonian program specifically in preference to the mechanical philosophy, which he regarded with suspicion. A famous comment from his *History of England* bolsters this interpretation:

> *While Newton seemed to draw off the veil from some of the mysteries of nature, he showed at the same time the imperfections of the mechanical philosophy; and thereby restored her ultimate secrets to that obscurity, in which they ever did and ever will remain. (Hume 1854 [1754–61]: vol. 5: 374)*

The question of whether to accept, and of how to interpret, absolute space, time and motion, and the related question of how to conceive of the relation between Newton's work in natural philosophy and the flourishing Leibnizian-Wolffian metaphysics on the Continent, continued to drive conversations in the middle of the eighteenth century. Just a few years after Châtelet published her *Institutions*, the mathematician Leonard Euler presented a novel approach to these two questions in a short paper entitled “Reflexions sur l'Espace et le Temps”, first published in 1748 in the *Mémoires de l'Académie des Sciences de Berlin*. The Berlin Academy had been witnessing a vociferous debate between Wolffians and Newtonians since 1740 (a debate that would continue until roughly 1759), one in which Euler played a role. Whereas one might regard the British philosophers, especially Berkeley and Hume, as arguing that philosophical principles and commitments take a kind of precedence in driving one's interpretations of the concepts of force, motion, space and time, Euler argued that natural philosophy—specifically, mechanics—ought to take precedence. The famous first sentence of his essay indicates why: he contends that the principles of mechanics—for instance, the principle of inertia—are so well established that it would be foolish to doubt them (Euler 1748: 324). In particular, if one's metaphysical commitments stand in tension with the concepts of space and motion found in geometry and mechanics, then one must adjust those commitments accordingly.

Kants interpretation

Kant began grappling with Newtonian ideas at the very beginning of his career—he discussed the inverse-square law in his first publication (Kant 1747: § 10)—and they would remain central both to his magnum opus, the *Critique of Pure Reason* (Kant 1787/1992) and to his *Metaphysical Foundations of Natural Science* (1786/2002). Early in the so-called pre-critical period, Kant diverged sharply from the approach toward natural philosophy defended by many Leibnizians in German-speaking Europe by deciding to accept the Newtonian theory of universal gravity, along with corresponding aspects of the Newtonian conception of matter, as a starting point for philosophical theorizing (Friedman 2012: 485–6). He makes this explicit already in 1763, in *The Only Possible Argument*:

I will attempt to provide an explanation of the origin of the world system according to the general laws of mechanics, not an explanation of the entire natural order, but only of the great masses of matter and their orbits, which constitute the most crudest foundation of nature … I will presuppose the universal gravitation of matter according to Newton or his followers in this project. If there are any who believe that through a definition of metaphysics formulated according to their own taste they can annihilate the conclusions established by men of perspicacity on the basis of observation and by means of mathematical inference—if there are such persons, they can skip the following propositions as something which has only a remote bearing on the main aim of this essay. (Kant 1763: AK 2: 139)

A rare case of Kantian irony, it seems. Already in this early text, Kant has clearly broken with his predecessors both in England and on the Continent, who insisted on disputing Newton's theory of universal gravity, either on metaphysical or theological grounds. Instead, Kant's work will be predicated on that theory. But Kant never became an orthodox Newtonian, any more than an orthodox Leibnizian (or Wolffian). This is evident from the radically different fates of two classic Newtonian concepts within the Kantian system: the idea that the theory of universal gravity shows that gravity is a feature of material bodies, along with the related concept of action at a distance, on the one hand; and absolute space, on the other. The quotation from 1763 above indicates that Kant was willing to endorse Newton's theory of universal gravity, despite the many objections raised against it by his Leibnizian predecessors. Indeed, he was also willing to accept the most radical interpretation of that theory, one according to which every material body in the world should be understood as bearing a feature called gravity, one that involves that body in actions at a distance on all other such bodies. As Kant puts it dramatically in Proposition 7 of the second chapter of *Metaphysical Foundations of Natural Science*: “The attraction essential to all matter is an immediate action of matter on other matter through empty space” (Kant 1786/2002: 223; AK 4: 512).

In the *Critique of Pure Reason*, for instance, Kant expressed a basically Leibnizian sympathy by arguing that there are fundamental metaphysical (and perhaps epistemic) difficulties with thinking of space as existing independently of all objects and all possible relations among them, as “actual entities “(*wirkliche Wesen*—A23/B37) in their own right. He does so in a passage that (perhaps confusingly) characterizes the Leibnizians as also defending a kind of realism about space, but we can focus solely on his criticism of the Newtonians:

Those, however, who assert the absolute reality of space and time, whether they assume it to be subsisting or only inhering, must themselves come into conflict with the principles of experience. For if they decide in favor of the first (which is generally the position of the mathematical investigators of nature), then they must assume two eternal and infinite self-subsisting non-entities *(space and time), which exist (yet without there being anything real) only in order to comprehend everything real within themselves. (A39/B56)*

If one regards space (like time) as existing independently of all objects and all possible relations, and yet one admits that space is causally inert and imperceptible, as one presumably must in the late eighteenth century, then one is committed to the idea that there is a kind of infinite and eternal non-entity in the world. Space is a kind of non-entity, Kant suggests, because on the one hand it is said to exist independently of everything else, and yet on the other hand, it is said to be causally inert and imperceptible, which would distinguish it from every other sort of thing that exists.

Conclusion

Newtonian theory of motion as attributing a special force or nature to material objects, there is an interpretation of Newton that is consistent with the new theory of ideas. He writes (Berkeley 1992: *De Motu*, §6):

Again, force, gravity, and terms of that sort are more often used in the concrete (and rightly so) so as to connote the body in motion, the effort of resisting, etc. But when they are used by philosophers to signify certain natures carved out and abstracted from all these things, natures which are not objects of sense, nor can be grasped by any force of intellect, nor pictured by the imagination, then indeed they breed errors and confusion.

There is little doubt, then, that the new British philosophy represented by Locke, Berkeley, and Hume in the early-to-mid eighteenth century was concerned to present interpretations of Newton's work that were consistent with their overarching philosophical commitments, principles and methods, or to alter those commitments, principles and methods as necessary.

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