The b-Chromatic Number of $L(CH_n)$ and $M(CH_n)$

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Abstract : The helm graph H_n is the graph obtained from wheel graph $W_n = C_n + K_1$ by adjoining a pendent edge node of the cycle while the closed helm is the graph obtained by joining each pendent vertex to form a cycle. In this paper we find the b-chromatic number of line graph of closed helm $L(CH_n)$ and Middle graph $M(CH_n)$.

Keywords: Proper colouring, b-colouring, b-chromatic number, Closed helm graph, Line graph, Middle graph.

I. INTRODUCTION

A b-coloring by k -colors is a proper coloring of the vertices of a graph G such that in each color class there exists a vertex which has neighbors in all the other $k-1$ color classes. The b-chromatic number $\varphi(G)$ is the largest integer k such that G admits a b-coloring with k -colors. The concept of b-coloring was introduced by Irving and Manlove [3] in 1999 and showed that the problem of determining b-chromatic number is NP-hard for general graphs but it is polynomial for trees. The upper bounds for the b-chromatic number were investigated in the work of Kouider M and Maheo M [5]. J.Vernold Vivin, M. Venkatachalam [9] investigated the b-chromatic number of corona graphs.The b-chromatic number of helm and closed helm graph were examined by Vaidya S K and Shukla M S [11]. Nadeem Ansari and Chandel R S and Rizwana Jamal [15] find out the b-chromatic number of Prism graph families. In this paper we examine the the b-chromatic number of line graph $L(G)$, Middle graph $M(G)$ of closed helm graph.

II. PRELIMINARIES

2.1 Line Graph

The line graph $L(G)$ [14] of a graph G is the graph whose vertex set is $E(G)$ and two vertices are adjacent in $L(G)$ whenever they are incident in G .

2.2 Middle Graph

The middle graph $M(G)$ [14] of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent if and only if either they are adjacent edges of G or one is a vertex of G and the other is an edge incident on it.

III. MAIN RESULTS

3.1 Theorem

For the line graph of closed helm graph, $\varphi[L(CH_n)] = n, n > 6$.

Proof:

A closed helm graph CH_n is the graph obtained from a cycle from a helm graph H_n by joining each pendent vertex to form a cycle. Let us consider the line graph of closed helm $L(CH_n), n > 6$. The vertex set of the line graph of closed helm can be partitioned as follows:

- The vertices introduced in the rim edges of CH_n form a complete graph K_n and the vertex set be $\{u_1, u_2, u_3, \ldots, u_n\}$,
- The vertices introduced in the inner cycle of CH_n denoted as $\{v_1, v_2, v_3, \ldots, v_n\}$,
- The vertices introduced in the pendent edges of H_n denoted as $\{w_1, w_2, w_3, \ldots, w_n\}$,
- And the vertices introduced in the outer cycle of CH_n denoted as $\{x_1, x_2, x_3, \ldots, x_n\}$.

The vertex v_1 is adjacent with v_2, v_n, u_1, u_2 and w_1, w_2 .

The vertex v_n is adjacent with v_1, v_{n-1}, u_1, u_n and w_1, w_n .

The remaining vertices in the inner cycle v_i for $i = 2,3,4,\ldots,n-1$, are adjacent with $v_{i-1}, v_{i+1}, u_i, u_{i+1}$ and w_i, w_{i+1}.

The vertex w_1 is connected with v_1, v_n, x_1, x_n and u_1 .

The vertex w_n is connected with $v_n, v_{n-1}, x_n, x_{n-1}$ and u_n .

The remaining verities introduced in the pendent edges of helm are w_i for $i = 2,3,4,\ldots,n-1$, adjacent with $v_{i-1}, v_i, x_{i-1}, x_i$ and u_i .

Now the vertex introduced in the outer cycle of closed helm x_1 is adjacent with w_1, w_2 and x_2, x_n .

Vertex x_n is adjacent with w_1, w_n and x_1, x_{n-1} .

The remaining vertices x_i for $i = 2,3,4,\ldots,n-1$, is adjacent with w_i, w_{i+1} and x_{i-1}, x_{i+1} .

The line graph of closed helm graph contains a complete graph K_n . Assigning the colours to $L(CH_n)$ as follows. First we color the complete graph K_n with n colours as $\{c_1, c_2, \ldots, c_n\}$. Now introduce a new colour c_{n+1} to the remaining vertices of the line graph of closed helm graph. Introducing a new colour to these vertices cannot harmonizes the new colour c_{n+1} . Since the vertices introduced in the inner cycle of closed helm v_i for $i=1,2,3,\ldots,n$ having maximum degree of 6 and the vertices introduced in the pendent edges of helm w_i for $i=1,2,3,\ldots,n$ having maximum degree of 5 and the vertices introduced in the outer cycle of closed helm x_i for $i=1,2,3,\ldots,n$ having maximum degree of 4. There fore c_n is the maximum colouring for $L(CH_n)$. Hence the b-chromatic number line graph of closed helm graph is $n, n>6$.

3.2 Theorem

For the Middle graph of closed helm graph, $\varphi[M(CH_n)]=n+1, n>6$.

Proof:

Let us consider the middle graph of closed helm $M(CH_n), n>6$. The vertex set of middle graph of closed helm can be portioned as follows:

- The apex vertex u and vertices introduced in the rim edges of CH_n form a complete graph K_{n+1} and the vertex set be $\{u,u_1,u_2,u_3,\ldots,u_n\}$,
- The vertices introduced in the inner cycle of CH_n denoted as $\{v_1,v_2,v_3,\ldots,v_{2n}\}$,
- The vertices introduced in the pendent edges of H_n denoted as $\{w_1,w_2,w_3,\ldots,w_n\}$,
- And the vertices introduced in the outer cycle of CH_n denoted as $\{x_1,x_2,x_3,\ldots,x_{2n}\}$.

The vertex v_1 is adjacent with v_2, v_{2n} and u_1, w_1.
The vertex v_{2n} is adjacent with $v_1, v_2, v_{2n-1}, v_{2n-2}$ and w_1, w_n, u_1, u_n.
Vertex v_2 is adjacent with v_1, v_3, v_4, v_{2n} and w_1, w_2, u_1, u_2.
Vertex v_{2n-1} is adjacent with $v_{2n}, v_{2n-2}, v_4, v_{2n}$ and w_n, u_n.
The remaining vertices in the inner cycle v_i for $i=4,6,8,\ldots,2n-2$, are adjacent with $v_{i-1}, v_{i+1}, v_{i-2}, v_{i+2}$ and vertices introduced in pendent edge of helm and in K_{n+1}.
The vertices v_j for $j=3,5,7,\ldots,2n-3$, are adjacent with v_{i-1}, v_{i+1} and vertices introduced in pendent edges of helm and in K_{n+1}.
The vertex w_1 is connected with v_1, v_2, v_{2n}, x_1, x_2, x_{2n} and u_1.
The vertex w_n is connected with $v_{2n}, v_{2n-1}, v_{2n-2}$, $x_{2n}, x_{2n-1}, x_{2n-2}$ and u_n.
The remaining verities introduced in the pendent edges of helm are w_i for $i=2,3,4,\ldots,n-1$, adjacent with v_i, v_{i+1}, v_{i+2}, x_i, x_{i+1}, x_{i+2} and u_n.
Now the vertex introduced in the outer cycle of closed helm x_1 is adjacent with x_2, x_{2n} and w_1.
Vertex x_{2n} is adjacent with x_1, x_{2n-1} and w_1, w_n.
The remaining vertices x_i for $i=2,3,4,\ldots,2n-1$, is adjacent with x_i, x_{i+1} and the vertices introduced in the pendent vertices.

The middle graph of closed helm graph contains a complete graph K_{n+1}. Assigning the colours to $M(CH_n)$ as follows. First we colour the complete graph K_{n+1} with with $n+1$ colours as $\{c_1,c_2,\ldots,c_{n+1}\}$. Now introduce a new colour c_{n+2} to the remaining vertices of the middle graph of closed helm graph. Introducing a new colour to these vertices cannot harmonizes the new colour c_{n+1}. Since the vertices introduced in the inner cycle of closed helm v_i for $i=1,3,5,\ldots,2n-1$ having maximum degree of 4 and the vertices v_j for $j=2,4,6,\ldots,2n$ having maximum degree of 8 the vertices introduced in the pendent edges of helm w_i for $i=1,3,5,\ldots,n$ having maximum degree of 7 and the vertices introduced in the outer cycle of closed helm x_i for $i=1,3,5,\ldots,2n-1$ having maximum degree of 3. The vertices for $i=2,4,6,\ldots,2n$ having maximum degree of 6. There fore c_{n+1} is the maximum colouring for $M(CH_n)$. Hence the b-chromatic number of middle graph of closed helm graph is $n+1, n>6$.

IV. RESULTS

- For the line graph of closed helm graph, $\varphi[L(CH_n)]=n, n>6$.
- For the Middle graph of closed helm graph, $\varphi[M(CH_n)]=n+1, n>6$.

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