Bianchi Type –I Viscous Fluid Cosmological Model in Wesson's Theory of Gravitation

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Abstract: *In this paper, Bianchi Type –I Cosmological Model is investigated in Wesson's scale invariant theory of gravitation. The matter field is considered in the form of viscous fluid. The field equations for scale invariant theory have been solved by using the equation of state. The gauge function depends on time coordinate only (Dirac gauge). Some physical and kinematical properties of the models are also discussed. It is found that the model admits viscous fluid distribution in scale invariant theory of gravitation.*

Key words*: Spherically symmetric space-time, Gauge function, Viscous fluid, Scale invariant theory*

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1. Introduction

Investigation of Bianchi type plays a vital role in the description of early stages of evolution of the universe. In particular Bianchi type –I present a simple picture of spatially homogeneous models of the universe. Mohanty and Daud [19] have studied Bianchi type –I cosmological model with gauge function in vacuum wherein they have shown that the model reduces to Kasner [7] one, when cosmological constant is zero and the model isotropizes as in Einstein's theory for a nonzero cosmological constant. Mishra [10] has obtained static plane symmetric cosmological models filled with perfect fluid in Wesson's [21, 22] theory of gravitation and also has studied the Bianchi type I cosmological model in this theory.

Beesham [3, 4, 5], Mohanty and Mishra [17, 18], Mishra [11, 12], Khadekar and Avachar [8], Mishra and Sahoo [13, 14, 15] have investigated several aspects of scale invariant theory. Recently Mishra et al. [16] has constructed Bianchi type III space-time in scale invariant theory with dark energy.

As far as our information goes there is no work in Bianchi type –I model with viscous fluid in Wesson's scale invariant theory of gravitation. In the observable universe, viscosity plays an important role in galaxy formation and modified the nature of singularity ([Misner, [9], Belinski and Khalatnikov [2]). Moreover it increases high degree of isotropy as observed in cosmic microwave background radiation (Weinberg [20]).

With this fact in mind, in this paper, we have considered and studied Bianchi type –I space –time in the presence of imperfect fluid distribution (viscous fluid) with Dirac guage function $\beta = \beta(ct)$ in Wesson's scale invariant theory of gravitation.

In section 2, we obtain the field equations in Wesson's scale invariant theory of gravitation for Bianchi type –I space –time in the presence of viscous fluid distributions. In section 3, the field equations of Wesson's theory are solved to obtained Bianchi type –I viscous fluid model in scale invariant theory. Section 4 is devoted to the discussion of physical properties of the model, while section 5 contains some conclusions.

2. Metric and Field Equations

We consider Bianchi type –I metric with gauge function $\beta = \beta(ct)$ as (Ellis and McCallum, 1969 [6])

$$dS_W^2 = \beta^2 dS_E^2 \tag{1}$$

with

$$dS_E^2 = -c^2dt^2 + A^2dx^2 + B^2(dy^2 + dz^2), \tag{2}$$

where $A = A(t), B = B(t)$ and dS_W and dS_E are the intervals in Wesson and Einstein theories respectively.
The field equations for scale invariant theory (Wesson, [21, 22]) with Dirac gauge are

$$G_{ij} + 2\frac{\beta_{;ij}}{\beta} - 4\frac{\beta_{,i}\beta_{,j}}{\beta^2} + \left(g^{ab}\frac{\beta_{,a}\beta_{,b}}{\beta^2} - 2g^{ab}\frac{\beta_{;ab}}{\beta}\right)g_{ij} + \Lambda_0\beta^2 g_{ij} = -\kappa T_{ij}, \tag{3}$$

where

$$G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}, \tag{4}$$

Here G_{ij} is the usual Einstein tensor, T_{ij} is the energy momentum tensor, R_{ij} is the Ricci tensor and R is the R is the Ricci scalar. Also coma (,) and semicolon (;) denote partial and covariant differentiation respectively. The cosmological term Λg_{ij} of Einstein theory is now transformed to $\Lambda_0 \beta^2 g_{ij}$ in scale invariant theory with the dimensional constant Λ_0. G and $\kappa = \frac{8\pi G}{c^4}$ are respectively the Newtonian gravitational constant and Wesson gravitational constant.
The energy momentum tensor for imperfect fluid with bulk viscosity in the form

$$T_{ij}^{m} = (\overline{p_m} + \rho_m c^2) u_i u_j + \overline{p_m} g_{ij} \tag{5}$$

together with

$$g_{ij} u^i u^j = -1 \tag{6}$$

$$\overline{p_m} = p_m - \xi\theta \tag{7}$$

where $\theta = u^i_{;i}$ and u^i is the four velocity vector of the fluid. ρ_m , p_m and ξ are energy density, proper isotropic pressure and bulk viscosity of the fluid respectively.
The surviving components of the Einstein tensor (4) for the metric (2) are

$$G_{11} = \frac{A^2}{c^2}\left[2\frac{B_{44}}{B} + \frac{{B_4}^2}{B^2}\right] \tag{8}$$

$$G_{22} = G_{33} = \frac{B^2}{c^2}\left[\frac{A_{44}}{A} + \frac{A_4 B_4}{AB} + \frac{B_{44}}{B}\right] \tag{9}$$

$$G_{44} = -\left[2\frac{A_4 B_4}{AB} + \frac{{B_4}^2}{B^2}\right] \tag{10}$$

$$\theta = u^i_{;i} = \left[\frac{A_4}{A} + 2\frac{B_4}{B}\right] \tag{11}$$

Here $A_4 = \frac{dA}{dt}$, etc.
Using commoving coordinate system $u^i = \delta^i_4$ and the metric (1), the field equations (3) can be expressed as

$$G_{11} = -\kappa\bar{p}_m A^2 - \frac{A^2}{c^2}\left[2\frac{\beta_{44}}{\beta} - \frac{{\beta_4}^2}{\beta^2} + 4\frac{\beta_4}{\beta}\left(\frac{B_4}{B}\right) + \Lambda_0\beta^2 c^2\right] \tag{12}$$

$$G_{22} = G_{33} = -\kappa\bar{p}_m B^2 - \frac{B^2}{c^2}\left[2\frac{\beta_{44}}{\beta} - \frac{{\beta_4}^2}{\beta^2} + 2\frac{\beta_4}{\beta}\left(\frac{A_4}{A} + \frac{B_4}{B}\right) + \Lambda_0\beta^2 c^2\right] \tag{13}$$

$$G_{44} = -\kappa\rho_m c^4 + \left[3\frac{{\beta_4}^2}{\beta^2} + 2\frac{\beta_4}{\beta}\left(\frac{A_4}{A} + 2\frac{B_4}{B}\right) + \Lambda_0\beta^2 c^2\right] \tag{14}$$

In the usual way (Wesson, [21, 22]), Eq. (3) and Eqs. (12)-(14) suggest the definition of quantities $\bar{p}_v$ (vacuum pressure with bulk viscosity) and ρ_v (vacuum density) that involves neither the Einstein tensor of conventional theory nor the properties of conventional matter. These two quantities can be obtained as

$$2\frac{\beta_{44}}{\beta} - \frac{{\beta_4}^2}{\beta^2} + 4\frac{\beta_4}{\beta}\left(\frac{B_4}{B}\right) + \Lambda_0\beta^2 c^2 = \kappa\bar{p}_v c^2 \tag{15}$$

$$2\frac{\beta_{44}}{\beta} - \frac{{\beta_4}^2}{\beta^2} + 2\frac{\beta_4}{\beta}\left(\frac{A_4}{A} + \frac{B_4}{B}\right) + \Lambda_0\beta^2 c^2 = \kappa\bar{p}_v c^2 \tag{16}$$

$$3\frac{{\beta_4}^2}{\beta^2} + 2\frac{\beta_4}{\beta}\left(\frac{A_4}{A} + 2\frac{B_4}{B}\right) + \Lambda_0\beta^2 c^2 = -\kappa\rho_v c^4 \tag{17}$$

where

$$\bar{p}_v = p_v - \xi\theta. \tag{18}$$

Here the quantities p_v and ρ_v are vacuum pressure and vacuum density respectively.
When there is no mater and the gauge function β is a constant. Eqs. (15) – (17) give the relation

$$c^2\rho_v = \frac{-c^4\lambda_{GR}}{8\pi G} = -\bar{p}_v, \text{ i.e. } c^2\rho_v + \bar{p}_v = 0 \tag{19}$$

which can be considered as the equation of state for vacuum. Here $\lambda_{GR} = \lambda_0\beta^2 =$ constant and thus it is the cosmological constant in general relativity and $\kappa = 8\pi G$. Also $\bar{p}_v$ is dependent on constants λ_{GR}, G and c and hence it is uniform in all directions. Thus $\bar{p}_v$ is isotropic in nature and consistent only when

$$A = k_1 B \tag{20}$$

where k_1 is a constant of integration. Without loss of generality, we take $k_1 = 1$.
Using Eqs. (11), (18) and (20) in (15)-(17) the pressure and energy density for the vacuum case can be obtained as

$$p_v = \frac{1}{\kappa c^2}\left[2\frac{\beta_{44}}{\beta} - \frac{{\beta_4}^2}{\beta^2} + 4\frac{\beta_4}{\beta}\left(\frac{A_4}{A}\right) + \Lambda_0\beta^2c^2 + 3\xi c^2\kappa\frac{A_4}{A}\right] \tag{21}$$

$$\rho_v = -\frac{1}{\kappa c^4}\left[6\frac{\beta_4}{\beta}\left(\frac{A_4}{A}\right) + 3\frac{{\beta_4}^2}{\beta^2} + \Lambda_0\beta^2c^2\right] \tag{22}$$

where p_v and ρ_v relate to the properties of vacuum only in conventional physics. Following Wesson (21, 22), the total pressure p_t and total density ρ_t can be defined as

$$\bar{p}_t = \bar{p}_m + \bar{p}_v = p_t - \xi\theta = (p_m - \xi\theta) + (p_v - \xi\theta) \Rightarrow p_t = p_m + p_v - \xi\theta \tag{23}$$

$$\rho_t = \rho_m + \rho_v \tag{24}$$

Using the aforesaid definition of p_t and ρ_t, components of Einstein tensor [Eqs. (8)- (10)] with Eq. (11) and the consistency condition (20), the field Eqs. (12)-(14) can be written in the following explicit form

$$2\frac{A_{44}}{A} + \frac{{A_4}^2}{A^2} = -\kappa p_t c^2 + 3\kappa\xi c^2\left(\frac{A_4}{A}\right) \tag{25}$$

$$\frac{{A_4}^2}{A^2} = \kappa\rho_t c^4 \tag{26}$$

3. Solutions of Field Equations

Eqs. (25) and (26) are two field equations with four unknowns p_t, ρ_t, A and ξ.
Hence to solve the system we consider the equation of state

$$p_{t} = \frac{1}{3}\rho_t c^2 \tag{27}$$

and take a relation (Banerjee et al [1])

$$\xi = \xi_0\frac{A_4}{A}, \tag{28}$$

where ξ_0 is a constant.
Use the Eqs. (27) and (28), Eqs. (25)-(26) gives,

$$\frac{A_{44}}{A} + \alpha\frac{A_4}{A} = 0$$

where $\alpha = 1 - \frac{3}{2}\xi_0\kappa c^2 =$ constant, which on integration gives

$$A = [(\alpha + 1)(c_1 t + c_2)]^{1/\alpha+1}, \tag{29}$$

where c_1 and c_2 are constants of integration.
Then in view of Eqs. (20), we have

$$A = B = [(\alpha + 1)(c_1 t + c_2)]^{1/\alpha+1} \tag{30}$$

The total pressure p_t and energy density ρ_t can be obtained as,

$$p_t = \frac{\rho_t c^2}{3} = \frac{1}{\kappa c^2}\left[\frac{c_1}{(\alpha+1)(c_1 t + c_2)}\right]^2 \tag{31}$$

Considering Dirac gauge function in the form $\beta = \frac{1}{ct}$ the vacuum pressure and vacuum density can be obtained as

$$p_v = \frac{1}{\kappa c^2}\left[\frac{\Lambda_0+3}{t^2} - \frac{4c_1}{(\alpha+1)t(c_1t+c_2)} + \frac{3c^2\xi_0\kappa c_1{}^2}{(\alpha+1)^2(c_1t+c_2)^2}\right], \tag{32}$$

$$\rho_v = -\frac{1}{\kappa c^4}\left[\frac{\Lambda_0+3}{t^2} - \frac{6c_1}{(\alpha+1)(c_1t+c_2)}\right] \tag{33}$$

and the matter pressure and density can be obtained as

$$p_m = \frac{1}{\kappa c^2}\left[\frac{4c_1}{(\alpha+1)t(c_1t+c_2)} + \frac{(1-3c^2\kappa\xi_0)c_1{}^2}{[(\alpha+1)(c_1t+c_2)]^2} - \frac{\Lambda_0+3}{t^2}\right], \tag{34}$$

$$\rho_m = -\frac{1}{\kappa c^4}\left[\frac{6c_1}{(\alpha+1)t(c_1t+c_2)} - \frac{3c_1{}^2}{[(\alpha+1)(c_1t+c_2)]^2} - \frac{\Lambda_0+3}{t^2}\right] \tag{35}$$

Thus the Bianchi –I model in scale invariant theory is given by

$$dS_W^2 = \frac{1}{c^2t^2}\left[-c^2dt^2 + [(\alpha+1)(c_1t+c_2)]^{2/\alpha+1}(dx^2+dy^2+dz^2)\right] \tag{36}$$

Using the transformation $t = e^T$ the above metric can be written as

$$dS_W^2 = -dT^2 + Q^2(T)[dx^2+dy^2+dz^2] \tag{37}$$

where

$$Q(T) = \frac{[(\alpha+1)(c_1e^T+c_2)]^{1/\alpha+1}}{ce^T}$$

4 Some Physical Properties

Now we study some physical properties of the model (37). The physical behavior of the model remains the same in the transformed time coordinate. i.e. $t(0,1,\infty) \to T(-\infty,1,\infty)$.
Also the new time coordinate covers the time region from past to future completely, so that we can have clear picture of the model.
The scalar expansion of the model (37) can be obtained as

$$\theta(T) = 3\frac{Q_T}{Q} = 3\left[\frac{c_1}{(\alpha+1)(c_1e^T+c_2)} - 1\right], \tag{38}$$

Here $Q_T = \frac{dQ}{dT}$

Thus we have

$$\theta(0) \to -3\left[\frac{\alpha c_1+(\alpha+1)c_2}{(\alpha+1)(c_1+c_2)}\right],$$

and $\theta \to -3$ as $T \to \infty$. Moreover at $T = log\left[\frac{c_1-c_2(\alpha+1)}{c_1(\alpha+1)}\right]$, the model ceases contraction for a moment. Thus the model contracts without admitting any singularity during evolution.
The shear scalar $\sigma = 0$ indicates that the shape of the universe is unchanged during the evolution. Also since $\frac{\sigma^2}{\theta^2} = 0$, the space-time is isotropized during evolution in this theory. As the acceleration is found to be zero, the matter particles follow geodesic path in this theory. The vorticity w of the model vanishes, which indicates that U^i is hyper surface orthogonal.
From Eq. (35), with proper choice of parameters, we get $\rho_m(0) =$ positive constant and $\rho_m \to 0$ as $T \to \infty$. Thus the universe starts evolving with constant matter density at initial epoch.
Also it has been observed that

$$\frac{\rho_m}{\theta^2} = \text{Constant at } T = 0 \text{ and } \frac{\rho_m}{\theta^2} = 0 \text{ at } T = \infty, \tag{39}$$

which confirms the homogeneity nature of the space-time during the evolution.
The spatial volume of the model (37) is found to be

$$V = \left[\frac{[(\alpha+1)(c_1e^T+c_2)]^{1/\alpha+1}}{ce^T}\right]^3 \tag{40}$$

$V \to \left[\frac{[(\alpha+1)(c_1+c_2)]^{1/\alpha+1}}{c}\right]^3$ as $T \to 0$ and $V \to 0$ as $T \to \infty$. So the universe starts with constant volume at initial epoch and expands with uniform rate till infinite future, where $V = 0$.

The Hubble parameter H for the model (37) is given by

$$H = \frac{Q_T}{Q} = \left[\frac{c_1}{(\alpha+1)(c_1 e^T + c_2)} - 1\right] \tag{41}$$

which determine the present rate of expansion of the universe. However, $H(0) =$ constant and $H \to -1$as $T \to \infty$, which indicates that the rate of expansion remains constant throughout the evolution.
Also the declaration parameter q for the model (37) can be calculated as

$$q = -\frac{Q_{TT}Q}{{Q_T}^2} = \left[1 - \left(\frac{c_1 + c_2 e^{-T}}{c_1(c_1 + c_2 e^{-T}) - (\alpha+1)e^{-T}}\right)^2\right] \tag{42}$$

As $T \to 0, q \to$constant, say L. This indicates that the expansion is speeding up for $L > 0$ and slowing down for $L < 0$.
For the period when $\left(\frac{c_1 + c_2 e^{-T}}{c_1(c_1 + c_2 e^{-T}) - (\alpha+1)e^{-T}}\right)^2 > 1$, we get an inflationary universe.

5. Conclusions

In this chapter, we have studied a Binachi type –I model in Wesson's scale invariant theory of gravitation. The model in this case starts evolving at the initial epoch with a constant volume and ends at an infinite future. Also the matter density ρ_m vanishes for $\Lambda_0 = -3\left[\frac{2-3\xi_0\kappa c^2}{4-3\xi_0\kappa c^2}\right]^2$ but $\rho_m \neq 0$ for $c_2 = 0$. This leads to unphysical situation. Thus for a viable physical situation one should have $\Lambda_0 \neq -3\left[\frac{2-3\xi_0\kappa c^2}{4-3\xi_0\kappa c^2}\right]^2$. Also the model in this case appears to be steady state.

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