# UPPER UNIDOMINATION NUMBER OF A PATH

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*Abstract :* The concept of unidominating function is introduced and the unidominating function of a path is studied in [6]. The concept of minimal unidominating function and upper unidomination number are introduced in [7]. In this paper the authors study the minimal unidominating functions of a path and determined its upper unidomination number. Further the number of minimal unidominating functions with maximum weight is found.

*IndexTerms* - **Path, unidominating function, minimal unidominating function, upper unidomination number**.

## I. Introduction

Graph theory is one of the most developing branches of Mathematics with wide applications to various branches of Science and Technology. Theory of domination in graphs introduced by Berg[1] and Ore [2] is a rapidly growing area of research in Graph Theory. Domination and its properties have been extensively studied by T.W.Haynes and others in [3, 4].
Hedetniemi et.al.[ 5 ] introduced the concept of dominating function and it is attracted by many researchers because of its applications. The concept of unidominating function is introduced by the authors and the unidominating functions of a path are studied in [6]. The concept of minimal unidominating function and upper unidomination number are introduced in [7].

In this paper minimal unidominating functions of a path are studied and the upper unidomination number of a path is found. Also the result on the number of minimal unidominating functions with maximum weight are obtained. Further the results obtained are illustrated.

## II. MINIMAL UNIDOMINATING FUNCTIONS AND UPPER UNIDOMINATION NUMBER:

The concepts of unidominating function, minimal unidominating function, upper unidomination number are defined as follows:

**Definition 2.1:** Let $G(V,E)$ be a graph. A function $f: V \to \{0,1\}$ is said to be a unidominating function

if $\sum_{u\in N[v]} f(u) \geq 1 \;\; \forall\, v \in V \text{ and } f(v) = 1,$

$\sum_{u\in N[v]} f(u) = 1 \;\; \forall\, v \in V \text{ and } f(v) = 0.$

where $N[v]$ is the closed neighbourhood of the vertex $v$.

**Definition 2.2:** Let $f$ and $g$ be functions from $V$ to $\{0,1\}$. We say that $f < g$ if $f(u) \leq g(u) \;\forall\, u \in V,$ with strict inequality for at least one vertex $u$.

**Definition 2.3:** A unidominating function $f: V \to \{0,1\}$ is called a minimal unidominating function if for all $g < f, g$ is not a unidominating function.

**Definition 2.4:** The upper unidomination number of a graph $G(V,E)$ is defined as $max\,\{f(V)/f \text{ is a minimal unidominating function}\},$

$where\; f(V) = \sum_{u\in V} f(u).$

The upper unidomination number of a graph G is denoted by $\Gamma_u(G)$.

## III. UPPER UNIDOMINATION NUMBER OF A PATH

In this section the upper unidomination number of a path and the number of minimal unidominating functions with maximum weight of a path are found.

**Theorem 3.1:** The upper unidomination number of a path $P_n$ is $\left\lfloor \frac{3n}{5} \right\rfloor$.

**Proof:** Let $P_n$ be a path with vertex set $V = \{v_1, v_2, \ldots\ldots, v_n\}$.
To find upper unidomination number of $P_n$, the following five cases arise.

**Case 1:** Let $n \equiv 0(\text{mod } 5)$.
Define a function $f: V \to \{0,1\}$ by

$$f(v_i) = \begin{cases} 1 & for\ i \equiv 2,3,4(mod\ 5), \\ 0 & for\ \ i \equiv 0,1(mod\ 5). \end{cases}$$

Now we check the condition of unidominating function at every vertex.

**Sub case 1:** Let $i \equiv 0(\text{mod } 5)$ and $i \neq n$. Then $f(v_i) = 0$.

Now $\sum_{u\in N[v_i]} f(u) = f(v_{i-1}) + f(v_i) + f(v_{i+1}) = 1 + 0 + 0 = 1.$

For $i = n$, $\sum_{u\in N[v_n]} f(u) = f(v_{n-1}) + f(v_n) = 1 + 0 = 1.$

**Sub case 2:** Let $i \equiv 1(mod\ 5)$ *and* $i \neq 1$. *Then* $f(v_i) = 0$.

Now $\sum_{u\in N[v_i]} f(u) = f(v_{i-1}) + f(v_i) + f(v_{i+1}) = 0 + 0 + 1 = 1.$

For $i = 1$, $\sum_{u\in N[v_1]} f(u) = f(v_1) + f(v_2) = 0 + 1 = 1.$

**Sub case 3:** Let $i \equiv 2(mod\ 5)$. *Then* $f(v_i) = 1$.

Now $\sum_{u\in N[v_i]} f(u) = f(v_{i-1}) + f(v_i) + f(v_{i+1}) = 0 + 1 + 1 = 2 > 1.$

**Sub case 4:** Let $i \equiv 3(mod\ 5)$.*Then* $f(v_i) = 1$.

Now $\sum_{u\in N[v_i]} f(u) = f(v_{i-1}) + f(v_i) + f(v_{i+1}) = 1 + 1 + 1 = 3 > 1.$

**Sub case 5:** Let $i \equiv 4(mod\ 5)$. *Then* $f(v_i) = 1$.

Now $\sum_{u\in N[v_i]} f(u) = f(v_{i-1}) + f(v_i) + f(v_{i+1}) = 1 + 1 + 0 = 2 > 1.$

Since $\sum_{u\in N[v_i]} f(u) \geq 1$ *when* $f(v_i) = 1$ *and* $\sum_{u\in N[v_i]} f(u) = 1$ *when* $f(v_i) = 0$,
it follows that $f$ is a unidominating function.
Now we check for the minimality of $f$.
Define a function $g: V \to \{0,1\}$ by
$g(v_i) = f(v_i)\ \forall\ v_i \in V, i \neq k, k \equiv 2(mod\ 5)$ and $g(v_k) = 0$.
Then by the definition of $f$ and $g$ it is obvious that $g < f$.
Suppose $k = 2$. Then

$$\sum_{u\in N[v_1]} g(u) = g(v_1) + g(v_2) = 0 + 0 = 0 \neq 1.$$

Suppose $k \neq 2$. Then

$$\sum_{u\in N[v_{k-1}]} g(u) = g(v_{k-2}) + g(v_{k-1}) + g(v_k) = 0 + 0 + 0 = 0 \neq 1.$$

Since $k \equiv 2(mod\ 5)$, $k - 1 \equiv 1(mod\ 5)$. Then $g(v_{k-1}) = f(v_{k-1}) = 0$.
Again $g(v_1) = f(v_1) = 0$.
That is $\sum_{u\in N[v]} g(u) \neq 1$ for which $g(v) = 0$.
This contradicts the definition of unidominating function.
Therefore $g$ is not a unidominating function.
Similarly when $k \equiv 3,4(mod\ 5)$, then also it can be shown that $g$ is not a unidominating function.
Since $g$ is defined arbitrarily, it follows that there exists no $g < f$ such that $g$ is a unidominating function. Hence for all possibilities of defining a function $g < f$, it can be seen that $g$ is not a unidominating function.
Hence $f$ is a minimal unidominating function.

Now $\sum_{u\in V} f(u) = \underbrace{0+1+1+1+0}_{} + \cdots + \underbrace{0+1+1+1+0}_{} = 3.\frac{n}{5} = \frac{3n}{5}.$

Therefore $\Gamma_u(P_n) \geq \frac{3n}{5} - - - (1)$

If $f$ is a minimal unidominating function of $P_n$, then it can be seen that amongst five consecutive vertices in $P_n$ at most three consecutive vertices can have functional value $1$ and at least two vertices must have functional value $0$.

Therefore sum of the functional values of five consecutive vertices is less than or equal to 3. That is

$$\sum_{i=1}^{5} f(v_i) \leq 3, \quad \sum_{i=6}^{10} f(v_i) \leq 3, \quad \ldots, \sum_{i=n-4}^{n} f(v_i) \leq 3.$$

Therefore $\sum_{u\in V} f(u) = \sum_{i=1}^{5} f(v_i) + \sum_{i=6}^{10} f(v_i) + \cdots + \sum_{i=n-4}^{n} f(v_i) \leq \underbrace{3+3+\cdots+3}_{\frac{n}{5}-times} \leq \frac{3n}{5}.$

This is true for any minimal unidominating function.
Therefore $\max\{f(V)/f$ is a minimal unidominating function$\} \leq \frac{3n}{5}$.
That is$\Gamma_u(P_n) \leq \frac{3n}{5} - - - (2)$
Thus from the inequalities$(1)$ and $(2)$, $\Gamma_u(P_n) = \frac{3n}{5} = \left\lceil\frac{3n}{5}\right\rceil$.

**Case 2:** Let $n \equiv 1(mod\ 5)$.
Define a function $f: V \to \{0,1\}$ by

$$f(v_i) = \begin{cases} 1 & for\ i \equiv 2,3,4(mod\ 5), i \neq n-2, \\ 0 & for\ i \equiv 0,1(mod\ 5), i \neq n, \end{cases}$$

and $f(v_{n-2}) = 0,\ f(v_n) = 1$. Then this function is defined similarly as the function $f$ defined in Case 1 except for the vertices $v_{n-2}$ and $v_n$. So we check the condition of unidominating function in the closed neighbourhood of $v_{n-3}, v_{n-2}, v_{n-1}$, and $v_n$.

$$\sum_{u\in N[v_{n-3}]} f(u) = f(v_{n-4}) + f(v_{n-3}) + f(v_{n-2}) = 1+1+0 = 2 > 1,$$

$$\sum_{u\in N[v_{n-2}]} f(u) = f(v_{n-3}) + f(v_{n-2}) + f(v_{n-1}) = 1+0+0 = 1,$$

$$\sum_{u\in N[v_{n-1}]} f(u) = f(v_{n-2}) + f(v_{n-1}) + f(v_n) = 0+0+1 = 1,$$

$$\sum_{u\in N[v_n]} f(u) = f(v_{n-1}) + f(v_n) = 0+1 = 1.$$

Since $\sum_{u\in N[v_i]} f(u) \geq 1$ when $f(v_i) = 1$ and $\sum_{u\in N[v_i]} f(u) = 1$ when $f(v_i) = 0$,

it follows that $f$ is a unidominating function.

Now we check for the minimality of $f$.

Define a function $g: V \to \{0,1\}$ by

$g(v_i) = f(v_i)\ \forall\, v_i \in V,\ \ i \neq n\ and\ \ g(v_n) = 0.$

Then by the definition of $f$ and $g$ it is obvious that $g < f$.

Now $g(v_n) = 0$, but $\sum_{u\in N[v_n]} g(u) = g(v_{n-1}) + g(v_n) = 0+0 = 0 \neq 1.$

Therefore $g$ is not a unidominating function.

Hence for all possibilities of defining a function $g < f$, it can be seen that $g$ is not a unidominating function.

Therefore $f$ is a minimal unidominating function.

Now $\sum_{u\in V} f(u) = \underbrace{0+1+1+1+0}+\cdots+\underbrace{0+1+1+0}+0+1 = \frac{3\,(n-6)}{5} + 2 + 1 \quad = \frac{3n-3}{5} = \left\lfloor \frac{3n}{5} \right\rfloor.$

(We will take 5 vertices as one group so that their functional values sum is 3 and there are $\frac{n-6}{5}$ such groups. The remaining vertices are 6 and their functional values sum is $2+1$).

Therefore $\Gamma_u(P_n) \geq \left\lfloor \frac{3n}{5} \right\rfloor$ --- (1)

Let $f$ be a minimal unidominating function of $P_n$.

Suppose $n = 6$. Then the possible assignment of functional values to these six verticesis $1{,}0{,}0{,}1{,}1{,}0$ $or$ $0{,}1{,}1{,}0{,}0{,}1$ $or$ $0{,}1{,}0{,}01{,}0$, so that $f(V) \leq 3$ $and$

$\Gamma_u(P_6) = 3 = \left\lfloor \frac{3n}{5} \right\rfloor = \left\lfloor \frac{18}{5} \right\rfloor.$

Let $n \geq 11$.

As in Case 1 of this theorem we have $\sum f(v_i) \leq 3$ for any five consecutive vertices.

Therefore $\sum_{i=2}^{n} f(v_i) \leq \frac{3(n-1)}{5}.$

Now we assign the functional value to $v_1$ as follows.

Suppose $f(v_1) = 0$.

Then $f(V) = f(v_1) + \sum_{i=2}^{n} f(v_i) \leq 0 + \frac{3(n-1)}{5} = \frac{3n-3}{5} = \left\lfloor \frac{3n}{5} \right\rfloor.$

Suppose $f(v_1) = 1$.

In such case among the $\frac{(n-1)}{5}$ sets of five consecutive vertices, there will be one set of five consecutive vertices whose functional values sum is 2. Otherwise the assignment of functional values makes $f$ no more a minimal unidominating function.

$$\text{Therefore}\quad f(V) = f(v_1) + \sum_{i=2}^{n-5} f(v_i) + \sum_{i=n-4}^{n} f(v_i)$$
$$\leq 1 + \frac{3(n-6)}{5} + 2 = \frac{3n-3}{5} = \left\lfloor \frac{3n}{5} \right\rfloor.$$

(Here this set need not be the last set of five consecutive vertices. It can be between the set of vertices $v_2, v_3, \ldots, v_{n-5}$. For convenience we have taken the last set of five consecutive vertices).

Since $f$ is arbitrary, it follows that $\Gamma_u(P_n) \leq \left\lfloor \frac{3n}{5} \right\rfloor$ --- (2)

Therefore from the inequalities (1) and (2), we have $\Gamma_u(P_n) = \left\lfloor \frac{3n}{5} \right\rfloor.$

**Case 3:** Let $n \equiv 2(mod\ 5)$.

Define a function $f: V \to \{0,1\}$ by

$$f(v_i) = \begin{cases} 1 & for\ \ i \equiv 2{,}3{,}4(mod\ 5), \\ 0 & for\ \ i \equiv 0{,}1(mod\ 5). \end{cases}$$

On similar lines to Case 1 we can show that $f$ is a minimal unidominating function.

Now $f(V)=\sum_{u\in V} f(u) = \underbrace{0+1+1+1+0}+\cdots+\underbrace{0\ +1+1+1+0}+0+1$

$$=\frac{3(n-2)}{5}+1=\frac{3n-1}{5}=\left\lfloor\frac{3n}{5}\right\rfloor.$$

By the definition of upper unidomination number,

$\Gamma_u(P_n)\geq\left\lfloor\frac{3n}{5}\right\rfloor$ --- (1)

Let $f$ be a minimal unidominating function of $P_n$.

Suppose $n=2$. Then the possibilities of assigning functional values to these two vertices is 1,0 or 0,1, so that $f(V)=1=\left\lfloor\frac{3n}{5}\right\rfloor=\left\lfloor\frac{6}{5}\right\rfloor$.

Let $n\geq 7$.

Now $n\equiv 2(mod\ 5)\Rightarrow n-2\equiv 0(mod\ 5)$. So by Case 1 we have

$$\sum_{i=1}^{n-2} f(v_i)\leq\frac{3(n-2)}{5}.$$

Then for the vertices $v_{n-1}$ and $v_n$, we have

$f(v_n)=0$ or $1$ and $f(v_{n-1})=1$ or $0$. So $f(v_{n-1})+f(v_n)=1$.

Therefore $\sum_{u\in V} f(u)=\sum_{i=1}^{n-2} f(v_i)+\big(f(v_{n-1})+f(v_n)\big)\leq\frac{3(n-2)}{5}+1=\frac{3n-1}{5}=\left\lfloor\frac{3n}{5}\right\rfloor$.

Thus $\Gamma_u(P_n)\leq\left\lfloor\frac{3n}{5}\right\rfloor$ --- (2)

Therefore from the inequalities (1) and (2), it follows that $\Gamma_u(P_n)=\left\lfloor\frac{3n}{5}\right\rfloor$.

**Case 4:** Let $n\equiv 3(mod\ 5)$.

Define a function $f:V\to\{0,1\}$ by

$$f(v_i)=\begin{cases}1 & for\ i\equiv 2,3,4(mod\ 5)\ and\ i\neq n,\\ 0 & for\ i\equiv 0,1(mod\ 5),\end{cases}$$

and $f(v_n)=0$.

We can verify in similar lines as in Case 1 that $f$ is a unidominating function.

Now we check for the minimality of $f$.

Define a function $g:V\to\{0,1\}$ by

$g(v_i)=f(v_i)\ \forall\ v_i\in V, i\neq n-1$ and $g(v_{n-1})=0$.

Then by the definition of $f$ and $g$ it is obvious that $g<f$.

Now $g(v_{n-1})=0$, but

$$\sum_{u\in N[v_{n-1}]} g(u)=g(v_{n-2})+g(v_{n-1})+g(v_n)=0+0+0=0\neq 1.$$

Therefore $g$ is not a unidominating function.

It can be seen that for all possibilities of defining a function $g<f$, $g$ is not a unidominating function.

Therefore $f$ is a minimal unidominating function.

Now $\sum_{u\in V} f(u)=\underbrace{0+1+1+1+0}+\cdots+\underbrace{0+1+1+1+0}+\underbrace{0+1+0}$

$$=\frac{3(n-3)}{5}+1=\frac{3n-4}{5}=\left\lfloor\frac{3n}{5}\right\rfloor.$$

Therefore $\Gamma_u(P_n)\geq\left\lfloor\frac{3n}{5}\right\rfloor$ --- (1).

Let $f$ be a minimal unidominating function of $P_n$.

Suppose $n=3$. Then the vertices $v_1,v_2,v_3$ have functional values 0,1,0 and this is the only one possibility, so that $\Gamma_u(P_3)=f(V)=1=\left\lfloor\frac{3n}{5}\right\rfloor=\left\lfloor\frac{9}{5}\right\rfloor$.

Let $n\geq 8$.

As in Case 1 of this theorem we have $\sum f(v_i)\leq 3$ for any five consecutive vertices.

Therefore $\sum_{i=2}^{n-2} f(v_i)\leq\frac{3(n-3)}{5}$.

Similar to Case 3, for the vertices $v_{n-1}$ and $v_n$, here also we have

$f(v_n)=0$ or $1$ and $f(v_{n-1})=1$ or $0$, so that $f(v_{n-1})+f(v_n)=1$.

Now we assign the functional values to $v_1$ as follows.

Suppose $f(v_1)=0$.

Then $f(V)=f(v_1)+\sum_{i=2}^{n-2} f(v_i)+\big(f(v_{n-1})+f(v_n)\big)\leq 0+\frac{3(n-3)}{5}+1=\frac{3n-4}{5}=\left\lfloor\frac{3n}{5}\right\rfloor$.

Suppose $f(v_1)=1$.

Then as in Case 2 we have

$$\sum_{i=2}^{n-2} f(v_i) = \sum_{i=2}^{n-7} f(v_i) + \sum_{i=n-6}^{n-2} f(v_i) \le \frac{3(n-8)}{5} + 2.$$

Therefore $f(V) = f(v_1) + \sum_{i=2}^{n-2} f(v_i) + \big(f(v_{n-1}) + f(v_n)\big)$

$$\le 1 + \frac{3(n-8)}{5} + 2 + 1 = \frac{3n-4}{5} = \left\lfloor \frac{3n}{5} \right\rfloor.$$

Since $f$ is arbitrary, it follows that $\Gamma_u(P_n) \le \left\lfloor \frac{3n}{5} \right\rfloor - - - (2)$

Therefore from the inequalities (1) and (2), it follows that $\Gamma_u(P_n) = \left\lfloor \frac{3n}{5} \right\rfloor$.

**Case 5:** Let $n \equiv 4(mod\ 5)$.

Define a function $f: V \to \{0,1\}$ by

$$f(v_i) = \begin{cases} 1 & for\ i \equiv 2,3,4(mod\ 5), and\ i \ne n, \\ 0 & otherwise. \end{cases}$$

and $f(v_n) = 0$.

Then on similar lines of Case 1 it can be shown that $f$ is a minimal unidominating function.

Further,

$$\sum_{u\in V} f(u) = \underbrace{0+1+1+1+0}+ - - - + \underbrace{0+1+1+1+0} + \underbrace{0+1+1+0} = \frac{3(n-4)}{5} + 2 = \frac{3n-2}{5} = \left\lfloor \frac{3n}{5} \right\rfloor.$$

Therefore $\Gamma_u(P_n) \ge \left\lfloor \frac{3n}{5} \right\rfloor - - - (1)$

Let $f$ be a minimal unidominating function.

Suppose $n = 4$. Then the possibilities of assigning functional values to these four vertices are 0,1,1,0 or 1,0,0,1, so that $f(V) = 2 = \left\lfloor \frac{3n}{5} \right\rfloor = \left\lfloor \frac{12}{5} \right\rfloor$.

Let $n \ge 9$.

If $f$ is any minimal unidominating function of $P_n$ then the pendent vertices $v_1$ and $v_n$ must satisfy the following conditions.

$f(v_1) + f(v_2) = 1$ $and$ $f(v_{n-1}) + f(v_n) = 1$.

Now $n \equiv 4(mod\ 5) \Rightarrow n - 4 \equiv 0(mod\ 5)$.Then asin Case 1,

$$\sum_{i=3}^{n-2} f(v_i) \le \frac{3(n-4)}{5}.$$

Therefore $f(V) = \sum_{u\in V} f(u) = f(v_1) + f(v_2) + \sum_{i=3}^{n-2} f(v_i) + \big(f(v_{n-1}) + f(v_n)\big)$

$$\le 1 + \frac{3(n-4)}{5} + 1 = \frac{3n-4}{5} = \left\lfloor \frac{3n}{5} \right\rfloor.$$

Since $f$ is arbitrary, it follows that $\Gamma_u(P_n) \le \left\lfloor \frac{3n}{5} \right\rfloor - - - (2)$

Therefore from the inequalities (1) and (2), $\Gamma_u(P_n) = \left\lfloor \frac{3n}{5} \right\rfloor$.

Thus for all possibilities of $n$, we have $\Gamma_u(P_n) = \left\lfloor \frac{3n}{5} \right\rfloor$. ∎

**Theorem 3.2:** The number of minimal unidominating functions of $P_n$ with maximum weight is

$$\begin{cases} 1 & \text{when } n \equiv 0(mod\ 5), \\ \left\lfloor \frac{2n}{5} \right\rfloor & \text{when } n \equiv 1(mod\ 5), \\ 2 & \text{when } n \equiv 2(mod\ 5), \\ \left\lceil \frac{n}{5} \right\rceil + \frac{1}{2}\left\lceil \frac{n}{5} \right\rceil \left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{5} \right\rfloor & \text{when } n \equiv 3(mod\ 5), \\ \left\lceil \frac{n}{5} \right\rceil + 1 & \text{when } n \equiv 4(mod\ 5). \end{cases}$$

**Proof:** Let $P_n$ be a path with vertex set $V = \{v_1, v_2, \dots, v_n\}$.

Now we find the number of minimal unidominating functions with maximum weight in the following five cases.

**Case 1:** Let $n \equiv 0(mod\ 5)$.

The function $f$ defined in Case 1 of Theorem 3.1 is given by


The functional values of $f$ are $0111001110 - - - 01110$.

Take $a = 01110$. Then the functional values of $f$ are in the pattern of $aaa \dots a$ (here there are $\frac{n}{5}$ $a's$). These letters $aaa \dots a$ can be arranged in one and only one way. Therefore there is one and only one minimal unidominating function with maximum weight.

**Case 2:** Let $n \equiv 1(mod\ 5)$.

The function $f$ defined in Case 2 of Theorem 3.1 is given by


The functional values of $f$ are $0111001110 - - - 01110011001$.
Take $a - 01110$, $b - 0110$. Then the functional values of $f$ are in the pattern of $aaa \ldots ab01$ (here there are $\frac{n-6}{5}$ $a's$). As there are $\frac{n-6}{5}$ $a's$ and one $b$, these letters can be arranged in $\frac{\left(\frac{n-6}{5}+1\right)!}{\left(\frac{n-6}{5}\right)!} = \frac{\left(\frac{n-1}{5}\right)!}{\left(\frac{n-6}{5}\right)!} = \frac{n-1}{5}$ ways.
Therefore there are $\frac{n-1}{5}$ minimal unidominating functions.
We further investigate some more minimal unidominating functions of $P_n$ with maximum weight in the following way.
Define a function $f_1 : V \to \{0,1\}$ by

$$f_1(v_i) = \begin{cases} 1 & for \quad i \equiv 0,1,4\ (mod\ 5),\ i \neq n, \\ 0 & for\ i \equiv 2,3 (mod\ 4), \end{cases}$$

and $f_1(v_n) = 0$.
First we show that $f_1$ is a unidominating function.
**Sub case 1:** Let $i \equiv 0 (mod\ 5)$ and $i \neq n-1$. Then $f_1(v_i) = 1$.

Now $\sum_{u\in N[v_i]} f_1(u) = f_1(v_{i-1}) + f_1(v_i) + f_1(v_{i+1}) = 1 + 1 + 1 = 3 > 1.$

For $i = n-1$, $\sum_{u\in N[v_{n-1}]} f_1(u) = f_1(v_{n-2}) + f_1(v_{n-1}) + f_1(v_n) = 1 + 1 + 0 = 2 > 1.$

**Sub case 2:** Let $i \equiv 1 (mod\ 5)$ and $i \neq 1, i \neq n$. Then $f_1(v_i) = 1$.

Now $\sum_{u\in N[v_i]} f_1(u) = f_1(v_{i-1}) + f_1(v_i) + f_1(v_{i+1}) = 1 + 1 + 0 = 2.$

For $i = 1$, $\sum_{u\in N[v_1]} f_1(u) = f_1(v_1) + f_1(v_2) = 1 + 0 = 1.$

For $i = n$, $\sum_{u\in N[v_n]} f_1(u) = f_1(v_{n-1}) + f_1(v_n) = 1 + 0 = 1.$

**Sub case 3:** Let $i \equiv 2 (mod\ 5)$. Then $f_1(v_i) = 0$.

Now $\sum_{u\in N[v_i]} f_1(u) = f_1(v_{i-1}) + f_1(v_i) + f_1(v_{i+1}) = 1 + 0 + 0 = 1.$

**Sub case 4:** Let $i \equiv 3 (mod\ 5)$. Then $f_1(v_i) = 0$.

Now $\sum_{u\in N[v_i]} f_1(u) = f_1(v_{i-1}) + f_1(v_i) + f_1(v_{i+1}) = 0 + 0 + 1 = 1.$

**Sub case 5:** Let $i \equiv 4 (mod\ 5)$. Then $f_1(v_i) = 1$.

Now $\sum_{u\in N[v_i]} f_1(u) = f_1(v_{i-1}) + f_1(v_i) + f_1(v_{i+1}) = 0 + 1 + 1 = 2 > 1.$

Thus $\sum_{u\in N[v_i]} f(u) \geq 1$ when $f(v_i) = 1$ and $\sum_{u\in N[v_i]} f(u) = 1$ when $f(v_i) = 0$.

Then it follows that $f_1$ is a unidominating function.
Now we check for the minimality of $f_1$.
Define a function $g : V \to \{0,1\}$ by
$g(v_i) = f_1(v_i)$ for $i = 1,2, \ldots, n, i \neq k$ for some $k \equiv 4 (mod\ 5)$,
and $g(v_k) = 0$.
Obviously $g < f_1$ and

$$\sum_{u\in N[v_{k-1}]} g(u) = g(v_{k-2}) + g(v_{k-1}) + g(v_k) = 0 + 0 + 0 = 0 \neq 1.$$

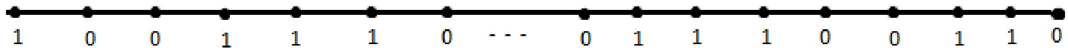
For $k = n-1$, $\sum_{u\in N[v_{n-2}]} g(u) = g(v_{n-3}) + g(v_{n-2}) + g(v_{n-1}) = 0 + 0 + 0 = 0 \neq 1.$

That is $\sum_{u\in N[v]} g(u) \neq 1$ for which $g(v) = 0$.

This contradicts the definition of unidominating function.
Therefore $g$ is not a unidominating function.
Similarly when $k \equiv 0,1 (mod\ 5)$ then also we can show that $g$ is not a unidominating function.
Therefore for all possibilities of defining a function $g < f_1$, it can be seen that $g$ is not a unidominating function.
Thus $f_1$ is a minimal unidominating function.
Further,

$$\sum_{u\in V} f_1(u) = 1 + 0 + \underbrace{0+1+1+1+0} + \cdots + \underbrace{0+1+1+1+0} + \underbrace{0+1+1+0} = 1 + 3\left(\frac{n-6}{5}\right) + 2 = \frac{3n-3}{5} = \left\lfloor \frac{3n}{5} \right\rfloor.$$

The functional values of $f_1$ are given by




The functional values of $f_1$ are $1001110 - - - 011100110$.
Take $a - 01110$, $b - 0110$.Then the functional values of $f_1$ are in the pattern of $10aaa \dots ab$ (here there are $\frac{n-6}{5}$ $a's$). In similar lines as above it can be proved that there exist $\frac{n-1}{5}$ minimal unidominating functions.
Thus there are $\frac{n-1}{5} + \frac{n-1}{5} = \frac{2n-2}{5} = \left\lfloor\frac{2n}{5}\right\rfloor$ minimal unidominating functions with maximum weight.

**Case 3**: Let $n \equiv 2(mod\ 5)$.
The function $f$ defined in Case 3 of Theorem 3.1 is given by


The functional values of $f$ are $01110 - - - 0111001$.
Take $a - 01110$. Then the functional values of $f$ are in the pattern of $aaa \dots a01$.
As these letters $aaa \dots a$ can be arranged in only one way, there exist one and only one minimal unidominating function.
Now as in Case 2, we will get another minimal unidominating function with the same weight.
Define a function $f_1 : V \to \{0,1\}$ by

$$f_1(v_i) = \begin{cases} 1 & for\ \ i \equiv 0,1,4\ (mod\ 5)\,, \\ 0 & otherwise. \end{cases}$$

On similar lines as in Case 2 of Theorem 3.2 we can show that $f_1$ is a minimal unidominating function.
Further,

$$\sum_{u \in V} f_1(u) = 1 + 0 + \underbrace{0+1+1+1+0} + \dots + \underbrace{0+1+1+1+0}\ = 1 + \frac{3(n-2)}{5} = \frac{3n-1}{5} = \left\lfloor\frac{3n}{5}\right\rfloor.$$

The functional values of $f_1$ are given by


The functional values of $f_1$ are $1001110 - - - 01110$.
Take $a - 01110$. Then the functional values of $f_1$ are in the pattern of $10aaa \dots a$.
These letters $aaa \dots a$ can be arranged in one and only one way. Therefore there exists only one function.
Thus there are two minimal unidominating functions with maximum weight.

**Case 4:** Let $n \equiv 3(mod\ 5)$.
The function $f$ defined in Case 4 of Theorem 3.1 is given by


The functional values of $f$ are $0111001110 - - - 01110010$.
Take $a - 01110$, $c - 010$. Then the functional values of $f$ are in the pattern of $aaa \dots ac$ (here there are $\frac{n-3}{5}$ $a's$). Then as in similar lines of Case 2 it can be seen that there are
$\frac{n-3}{5} + 1 = \frac{n+2}{5} = \left\lceil\frac{n}{5}\right\rceil$ minimal unidominating functions with maximum weight.
Now as in Case 2 and Case 3, we will get some other minimal unidominating functions with maximum weight.
Define another function $f_1 : V \to \{0,1\}$ by
$f_1(v_i) = f(v_i)\ \ \forall\, v_i \in V, for\ i \neq n-4, n-2,$
and $f_1(v_{n-4}) = 0\ \ and\ \ f_1(v_{n-2}) = 1,\ \ n \geq 8.$
Then we can check easily the condition of unidominating function in the closed neighbourhood of $v_{n-4}$, $v_{n-3}, v_{n-2}$ and $v_{n-1}$ and hence it follows that $f_1$ is a unidominating function and which is also minimal.
Then

$$\sum_{u \in V} f_1(u) = \underbrace{0+1+1+1+0} + \dots + \underbrace{0+1+1+1+0} + \underbrace{0+1+1+0} + \ \underbrace{0+1+1+0} = \frac{3(n-8)}{5} + 4 = \frac{3n-4}{5} = \left\lfloor\frac{3n}{5}\right\rfloor.$$

The function $f_1$ is given by


The functional values of $f_1$ are $01110 - - - 0111001100110$.
Take $a - 01110$, $b - 0110$. Then the functional values of $f_1$ are in the pattern of $aa \dots abb$. (here there are $\frac{n-8}{5}$ $a's$).
As there are $\frac{n-8}{5}$ $a's$ and two $b's$, these letters $a's$ and $b's$ can be arranged in
$\frac{\left(\frac{n-8}{5}+2\right)!}{\left(\frac{n-8}{5}\right)!2!} = \frac{\left(\frac{n+2}{5}\right)!}{\left(\frac{n-8}{5}\right)!2!} = \frac{1}{2}\left(\frac{n+2}{5}\right)\left(\frac{n-3}{5}\right) = \frac{1}{2}\left\lceil\frac{n}{5}\right\rceil\left\lfloor\frac{n}{5}\right\rfloor$ ways. Therefore there exist $\frac{1}{2}\left\lceil\frac{n}{5}\right\rceil\left\lfloor\frac{n}{5}\right\rfloor$ minimal unidominating functions.
Define another function $f_2 : V \to \{0,1\}$ by

$$f_2(v_i) = \begin{cases} 1 & for\ i \equiv 0,1,4\ (mod\ 5), i \neq n-2, \\ 0 & for\ i \equiv 2,3 (mod\ 5), i \neq n, \end{cases}$$

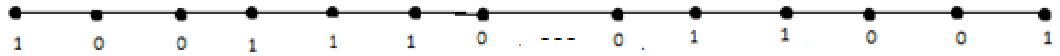
and $f_2(v_{n-2}) = 0,\ f_2(v_n) = 1,\ \ n \geq 8.$

Similar to earlier cases we can show that the function $f_2$ is a minimal unidominating function. Further,

$$\sum_{u \in V} f_2(u) = 1 + 0 + \underbrace{0+1+1+1+0} + \cdots + \underbrace{0+1+1+1+0} + \underbrace{0+1+1+0} + 0 + 1 = 1 + \frac{3(n-8)}{5} + 2 + 1 = \frac{3n-4}{5}$$

$$= \left\lfloor \frac{3n}{5} \right\rfloor.$$

The function $f_2$ is given by




The functional values of $f_2$ are $1001110 --- 01110011001.$

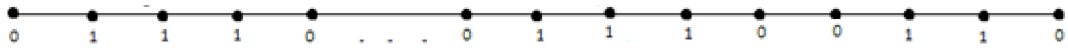
Take $a - 01110,\ b - 0110$.Then the functional values of $f_2$ are in the pattern of $10aaa \ldots ab01$. (here there are $\frac{n-8}{5}$ $a's$.)

As there are $\frac{n-8}{5}$ $a's$ and one $b$, there exists $\frac{n-8}{5} + 1 = \frac{n-3}{5} = \left\lfloor \frac{n}{5} \right\rfloor$ minimal unidominating functions.

Thus there are $\left\lceil \frac{n}{5} \right\rceil + \frac{1}{2} \left\lceil \frac{n}{5} \right\rceil \left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{5} \right\rfloor$ minimal unidominating functions with maximum weight.

**Case 5:** Let $n \equiv 4 (mod\ 5).$

The function $f$ defined in Case 4 of Theorem 3.1 is given by




The functional values of $f$ are $01110 --- 011100110.$

Take $a - 01110,\ b - 0110.$ Then the functional values of $f$ are in the pattern of $aa \ldots ab$ (here there are $\frac{n-4}{5}$ $a's$). On similar lines to Case 4 we can see that there are

$\frac{n-4}{5} + 1 = \frac{n+1}{5} = \left\lceil \frac{n}{5} \right\rceil$ minimal unidominating functions with maximum weight.

As in previous cases we investigate for another minimal unidominating function with maximum weight.

Define another function $f_1 : V \to \{0,1\}$ by

$$f_1(v_i) = \begin{cases} 1 & for\ i \equiv 0,1,4 (mod\ 5), \\ 0 & otherwise. \end{cases}$$

Similar to earlier cases we can show that $f_1$ is a minimal unidominating function.

Now

$$\sum_{u \in V} f_1(u) = 1 + 0 + \underbrace{0+1+1+1+0} + \cdots + \underbrace{0+1+1+1+0} + 0 + 1$$

$$= 1 + \frac{3(n-4)}{5} + 1 = \frac{3n-2}{5} = \left\lfloor \frac{3n}{5} \right\rfloor.$$

The function $f_1$ is given by


The functional values of $f_1$ are $1001110 --- 0111001.$

Take $a - 01110.$ Then the functional values of $f_1$ are in the pattern of $10aaa \ldots a01.$

As these letters $aa \ldots a$ can be arranged in only one way, there exists one and only one minimal unidominating function.

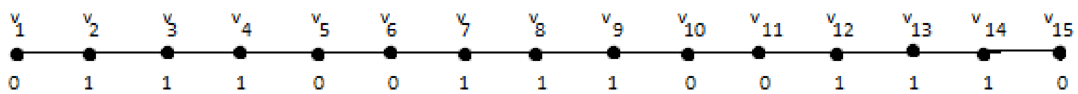
Thus there are $\left\lceil \frac{n}{5} \right\rceil + 1$ minimal unidominating functions with maximum weight.■

## IV. ILLUSTRATIONS

**Example 4.1:** Let $n = 15.$

Obviously 15$\equiv 0 (mod\ 5).$

The functional values of a minimal unidominating function $f$ defined in Case 1 of Theorem 3.1 are given at the corresponding vertices of $P_{15}$.



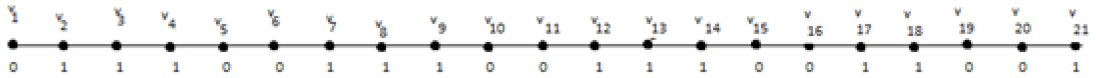

Upper unidomination number of $P_{15}$ is $\Gamma_u(P_{15}) = \left\lfloor \frac{45}{5} \right\rfloor = 9.$

There is only one minimal unidominating function for $P_{15}$ with maximum weight 9.■

**Example 4. 2:** Let $n = 21.$

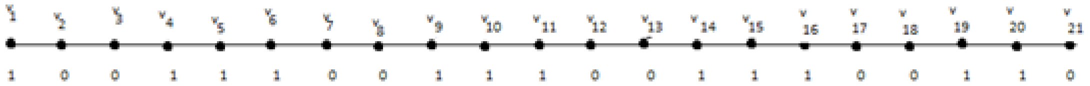
Clearly 21$\equiv 1 (mod\ 5).$

The functional values of a minimal unidominating function $f$ defined in Case 2 of Theorem 3.1 are given at the corresponding vertices of $P_{21}$.



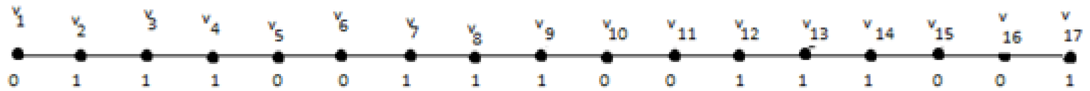

Upper unidomination number of $P_{21}$ is $\Gamma_u(P_{21}) = \left\lfloor \frac{63}{5} \right\rfloor = 12$.

There are 4 minimal unidominating functions that exists from $f$ with maximum weight12. The functional values of another minimal unidominating function$f_1$ defined in Case 2 of
Theorem 3.2 are given at the corresponding vertices of $P_{21}$.



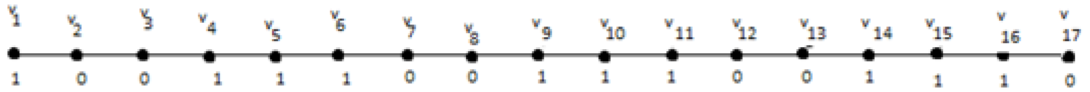

There are four such minimal unidominating functions.
Thus there are $\left\lfloor \frac{2x21}{5} \right\rfloor = 8 = (4+4)$ minimal unidominating functions with maximum weight 12.∎

**Example 4.3:** Let $n = 17$.
Clearly 17$\equiv 2(mod\ 5)$.
The functional values of a minimal unidominating function $f$ defined in Case 3 of
Theorem 3.1 are given at the corresponding vertices of $P_{17}$.



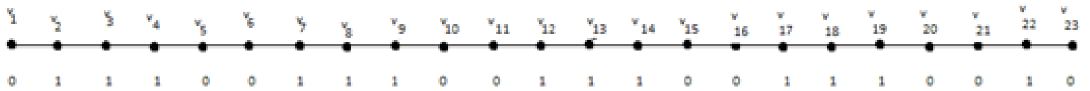

Upper unidomination number of $P_{17}$ is $\Gamma_u(P_{17}) = \left\lfloor \frac{51}{5} \right\rfloor = 10$.

There exists only one minimal unidominating function.
The functional values of another minimal unidominating function $f_1$ defined in
Case 3 of Theorem 3.2 are given at the corresponding vertices of $P_{17}$.




There exists only one minimal unidominating function.
Thus there are two minimal unidominating functions for $P_{17}$ with maximum weight10.∎

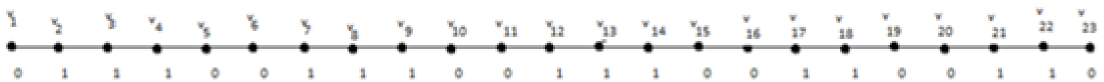
**Example 4.4:** Let $n = 23$.
We know that 23$\equiv 3(mod\ 5)$.
The functional values of a minimal unidominating function $f$ defined in Case 4 of
Theorem 3.1 are given at the corresponding vertices of $P_{23}$.




Upper unidomination number of $P_{23}$ is $\Gamma_u(P_{23}) = \left\lfloor \frac{69}{5} \right\rfloor$=13.

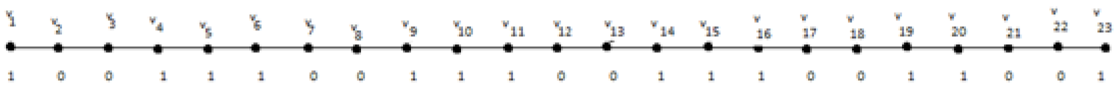
There are$\left\lceil \frac{23}{5} \right\rceil = 5$ minimal unidominating functions with maximum weight 13.

The functional values of another minimal unidominating function $f_1$ defined in
Case 4 of Theorem 3.2 are given at the corresponding vertices of $P_{23}$.




There are $\frac{1}{2}\left\lceil \frac{23}{5} \right\rceil \left\lfloor \frac{23}{5} \right\rfloor = 10$ minimal unidominating functions with maximum weight13.

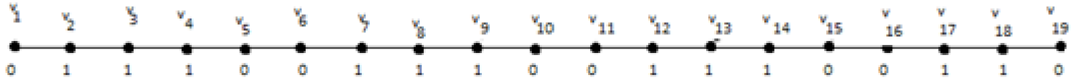
The functional values of another minimal unidominating function $f_2$ defined in
Case 4 of Theorem 3.2 are given at the corresponding vertices of $P_{23}$.




There are $\left\lfloor \frac{23}{5} \right\rfloor = 4$ minimal unidominating functions with maximum weight13.

Thus there are$5 + 10 + 4 = 19$ minimal unidominating functions for $P_{23}$ with maximum weight13.∎

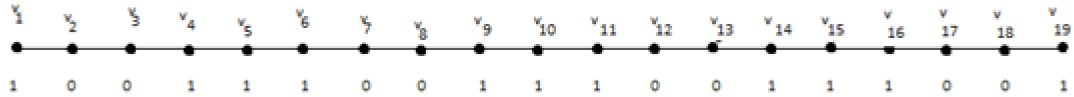
**Example 4.5:** Let $n = 19$.
Clearly $19 \equiv 4(mod\ 5)$.
The functional values of a minimal unidominating function $f$ defined in Case 5 of
Theorem 3.1 are given at the corresponding vertices of $P_{19}$.




Upper unidomination number of $P_{19}$ is $\Gamma_u(P_{19}) = \left\lfloor\frac{57}{5}\right\rfloor$=11.

There are $\left\lfloor\frac{19}{5}\right\rfloor = 4$ minimal unidominating functions with maximum weight11.

The functional values of another minimal unidominating function $f_1$ defined in Case 5 of Theorem 3.2 are given at the corresponding vertices of $P_{19}$.




There is one and only one minimal unidominating function with maximum weight 11.

Thus there are $\left\lfloor\frac{19}{5}\right\rfloor + 1 = 5$ minimal unidominating functions for $P_{19}$ with maximum weight 11.■

**V. CONCLUSION:** The upper unidomination number of a path is proved in five cases basing on the number of vertices. This work gives a scope to find upper unidomination number of a cycle and upper total unidomination number of a path.

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