Ricci Calculus on Tensor Fields

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Abstract

This paper attempts to study **Ricci calculus** constitutes the rules of index notation and manipulation for tensors and tensor fields on a differentiable manifold, with or without a metric **tensor field** or connection. It is also the modern name for what used to be called the absolute differential calculus (the foundation of tensor calculus), developed by Gregorio Ricci-Curbastro in 1887–1896, and subsequently popularized in a paper written with his pupil Tullio Levi-Civita in 1900. A component of a tensor is a real number that is used as a coefficient of a basis element for the tensor space. The tensor is the sum of its components multiplied by their corresponding basis elements. Tensors and tensor fields can be expressed in terms of their components, and operations on tensors and tensor fields can be expressed in terms of operations on their components. The description of tensor fields and operations on them in terms of their components is the focus of the Ricci calculus. This notation allows an efficient expression of such tensor fields and operations. While much of the notation may be applied with any tensors, operations relating to a differential structure are only applicable to tensor fields. Where needed, the notation extends to components of non-tensors, particularly multidimensional arrays.

A tensor may be expressed as a linear sum of the tensor product of vector and covector basis elements. The resulting tensor components are labelled by indices of the basis. Each index has one possible value per dimension of the underlying vector space. The number of indices equals the degree (or order) of the tensor. For compactness and convenience, the Ricci calculus incorporates Einstein notation, which implies summation over indices repeated within a term and universal quantification over free indices. Expressions in the notation of the Ricci calculus may generally be interpreted as a set of simultaneous equations relating the components as functions over a manifold, usually more specifically as functions of the coordinates on the manifold. This allows intuitive manipulation of expressions with familiarity of only a limited set of rules.

Key words: Ricci calculus, tensor fields, Einstein notation, differential structure

Introduction

An nth-rank tensor in m-dimensional space is a mathematical object that has n indices and m^n components and obeys certain transformation rules. Each index of a tensor ranges over the number of dimensions of space. However, the dimension of the space is largely irrelevant in most tensor equations (with the notable exception of the contracted Kronecker delta). Tensors are generalizations of scalars (that have no indices), vectors (that have exactly one index), and matrices (that have exactly two indices) to an arbitrary number of indices.

Tensors provide a natural and concise mathematical framework for formulating and solving problems in areas of physics such as elasticity, fluid mechanics, and general relativity.

The notation for a tensor is similar to that of a matrix (i.e., $\mathsf{A} = (a_{i\,j})$), except that a tensor $a_{i\,j\,k\,\ldots}$, $a^{i\,j\,k\,\ldots}$, $a_i{}^{j\,k}\,\ldots$, etc., may have an arbitrary number of indices. In addition, a tensor with rank $r+s$ may be of mixed type (r, s), consisting of r so-called "contravariant" (upper) indices and s "covariant" (lower) indices. Note that the positions of the slots in which contravariant and covariant indices are placed are significant so, for example, $a_{\mu\nu}{}^{\lambda}$ is distinct from $a_{\mu}{}^{\nu\lambda}$.

While the distinction between covariant and contravariant indices must be made for general tensors, the two are equivalent for tensors in three-dimensional Euclidean space, and such tensors are known as Cartesian tensors.

Objects that transform like zeroth-rank tensors are called scalars, those that transform like first-rank tensors are called vectors, and those that transform like second-rank tensors are called matrices. In tensor notation, a vector $\mathbf{v}$ would be written v_i, where $i = 1$, ..., m, and matrix is a tensor of type $(1, 1)$, which would be written $a_i{}^j$ in tensor notation.

Tensors may be operated on by other tensors (such as metric tensors, the permutation tensor, or the Kronecker delta) or by tensor operators (such as the covariant derivative). The manipulation of tensor indices to produce identities or to simplify expressions is known as index gymnastics, which includes index lowering and index raising as special cases. These can be achieved through multiplication by a so-called metric tensor $g_{i\,j}$, $g^{i\,j}$, $g_i{}^j$, etc., e.g.,

$$g^{i\,j} A_j = A^i \qquad (1)$$

$$g_{i\,j} A^j = A_i \qquad (2)$$

(Arfken 1985, p. 159).

Tensor notation can provide a very concise way of writing vector and more general identities. For example, in tensor notation, the dot product $\mathbf{u}\cdot\mathbf{v}$ is simply written

$$\mathbf{u}\cdot\mathbf{v} = u_i\, v^i, \qquad (3)$$

where repeated indices are summed over (Einstein summation). Similarly, the cross product can be concisely written as

$$(\mathbf{u}\times\mathbf{v})_i = \epsilon_{i\,j\,k}\, u^j\, v^k, \qquad (4)$$

where $\epsilon_{i\,j\,k}$ is the permutation tensor.

Contravariant second-rank tensors are objects which transform as

$$A'^{\,i\,j} = \frac{\partial x'_i}{\partial x_k}\,\frac{\partial x'_j}{\partial x_l}\, A^{k\,l}. \qquad (5)$$

Covariant second-rank tensors are objects which transform as

$$C'_{i\,j} = \frac{\partial x_k}{\partial x'_i}\,\frac{\partial x_l}{\partial x'_j}\, C_{k\,l}. \qquad (6)$$

Mixed second-rank tensors are objects which transform as

$$B'_j{}^i = \frac{\partial x'_i}{\partial x_k} \frac{\partial x_l}{\partial x'_j} B^k{}_l. \quad (7)$$

If two tensors A and B have the same rank and the same covariant and contravariant indices, then they can be added in the obvious way,

$$A^{ij} + B^{ij} = C^{ij} \quad (8)$$

$$A_{ij} + B_{ij} = C_{ij} \quad (9)$$

$$A^i{}_j + B^i{}_j = C^i{}_j. \quad (10)$$

The generalization of the dot product applied to tensors is called tensor contraction, and consists of setting two unlike indices equal to each other and then summing using the Einstein summation convention. Various types of derivatives can be taken of tensors, the most common being the comma derivative and covariant derivative.

If the components of any tensor of any tensor rank vanish in one particular coordinate system, they vanish in all coordinate systems. A transformation of the variables of a tensor changes the tensor into another whose components are linear homogeneous functions of the components of the original tensor.

A tensor space of type (r, s) can be described as a vector space tensor product between r copies of vector fields and s copies of the dual vector fields, i.e., one-forms. For example,

$$T^{(3,1)} = TM \otimes TM \otimes TM \otimes T^* M \quad (11)$$

is the vector bundle of $(3, 1)$-tensors on a manifold M, where TM is the tangent bundle of M and $T^* M$ is its dual. Tensors of type (r, s) form a vector space. This description generalized to any tensor type, and an invertible linear map $J : V \to W$ induces a map $\tilde{J} : V \otimes V^* \to W \otimes W^*$, where V^* is the dual vector space and J the Jacobian, defined by

$$\tilde{J}(v_1 \otimes v_2^*) = \left(J v_1 \otimes \left(J^T\right)^{-1} v_2^*\right), \quad (12)$$

where J^T is the pullback map of a form is defined using the transpose of the Jacobian. This definition can be extended similarly to other tensor products of V and V^*. When there is a change of coordinates, then tensors transform similarly, with J the Jacobian of the linear transformation.

Objective:

This paper intends to explore and analyze **Ricci calculus** as a formal system in which index notation is used to define tensors and tensor fields and the rules for their manipulation.

Ricci calculus , tensor components

While most of the expressions of the Ricci calculus are valid for arbitrary bases, the expressions involving partial derivatives of tensor components with respect to coordinates apply only with a coordinate basis: a basis that is defined through differentiation with respect to the coordinates. Coordinates are typically denoted by xμ, but do not in general form the components of a vector. In flat spacetime with linear coordinatization, a tuple of differences in coordinates, Δxμ, can be treated as a contravariant vector. With the same constraints on the space and on the choice of coordinate system, the partial derivatives with respect to the coordinates yield a result that is effectively covariant. Aside from use in this special case, the partial derivatives of components of tensors do not in general transform covariantly, but are useful in building expressions that are covariant, albeit still with a coordinate basis if the partial derivatives are explicitly used, as with the covariant

tensor having specific transformation properties (cf., a covariant tensor). To examine the transformation properties of a contravariant tensor, first consider a tensor of rank 1 (a vector)

$$d\,\mathbf{r} = d\,x_1\,\hat{\mathbf{x}}_1 + d\,x_2\,\hat{\mathbf{x}}_2 + d\,x_3\,\hat{\mathbf{x}}_3, \tag{1}$$

for which

$$d\,x_i' = \frac{\partial x_i'}{\partial x_j}\,d\,x_j. \tag{2}$$

Now let $A_i \equiv d\,x_i$, then any set of quantities A_j which transform according to

$$A_i' = \frac{\partial x_i'}{\partial x_j}\,A_j, \tag{3}$$

or, defining

$$a_{i\,j} \equiv \frac{\partial x_i'}{\partial x_j}, \tag{4}$$

according to

$$A_i' = a_{i\,j}\,A_j \tag{5}$$

is a contravariant tensor. Contravariant tensors are indicated with raised indices, i.e., a^{μ}.

Covariant tensors

tensor with differing transformation properties, denoted a_{ν}. However, in three-dimensional Euclidean space,

$$\frac{\partial x_j}{\partial x_i'} = \frac{\partial x_i'}{\partial x_j} \equiv a_{i\,j} \tag{6}$$

for $i, j = 1$, 2, 3, meaning that contravariant and covariant tensors are equivalent. Such tensors are known as Cartesian tensor. The two types of tensors do differ in higher dimensions, however.

Contravariant four-vectors satisfy

$$a^{\mu} = \Lambda^{\mu}_{\nu}\,a^{\nu}, \tag{7}$$

where Λ is a Lorentz tensor.

To turn a covariant tensor a_{ν} into a contravariant tensor a^{μ} (index raising), use the metric tensor $g^{\mu\nu}$ to write

$$g^{\mu\nu} a_\nu = a^\mu . \quad (8)$$

Covariant and contravariant indices can be used simultaneously in a mixed tensor.

Tensors are simply mathematical objects that can be used to describe physical properties, just like scalars and vectors. In fact tensors are merely a generalisation of scalars and vectors; a scalar is a zero rank tensor, and a vector is a first rank tensor.

The rank (or order) of a tensor is defined by the number of directions (and hence the dimensionality of the array) required to describe it. For example, properties that require one direction (first rank) can be fully described by a 3×1 column vector, and properties that require two directions (second rank tensors), can be described by 9 numbers, as a 3×3 matrix. As such, in general an n^{th} rank tensor can be described by 3^n coefficients.

The need for second rank tensors comes when we need to consider more than one direction to describe one of these physical properties. A good example of this is if we need to describe the electrical conductivity of a general, anisotropic crystal. We know that in general for isotropic conductors that obey Ohm's law:

$$\boldsymbol{j} = \sigma \boldsymbol{E}$$

Which means that the current density **j** is parallel to the applied electric field, **E** and that each component of **j** is linearly proportional to each component of **E**. (e.g. $\boldsymbol{j}_1 = \sigma \boldsymbol{E}_1$).

However in an anisotropic material, the current density induced will not necessarily be parallel to the applied electric field due to preferred directions of current flow within the crystal (a good example of this is in graphite). This means that in general each component of the current density vector can depend on all the components of the electric field:

Conclusion

Tensors are mathematical objects that can be used to describe physical properties, just like scalars and vectors. In fact tensors are merely a generalisation of scalars and vectors; a scalar is a zero rank tensor, and a vector is a first rank tensor.

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