SOME GEOMETRIC AGGREGATION OPERATORS BASED ON BIPOLAR PICTURE FUZZY SETS AND THEIR APPLICATION IN MULTIPLE ATTRIBUTE DECISION MAKING

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Abstract: In this paper, we introduce the Bipolar Picture Fuzzy set and their Relations. Also we develop Bipolar Picture Fuzzy Geometric operators and discuss their properties. We further propose a Bipolar Picture Fuzzy Multiple Attribute Decision making problems. A numerical example for the method is given to demonstrate the effectiveness and application of our proposed method.

Keyword: Bipolar Picture Fuzzy set, Bipolar Picture Fuzzy set Relations, Bipolar Picture Fuzzy Geometric operators.

I. INTRODUCTION

Zadeh [20] who introduced the fuzzy set in 1965 which was characterized by a membership degree of range [0, 1]. Intuitionistic Fuzzy set which was an extension of Fuzzy set given by Atanassov [1] in 1986. Intuitionistic Fuzzy set captured an association degree as well as non-association degree becomes the generalization of fuzzy set. The Characteristic of Intuitionistic fuzzy set is to assigns each element a Membership degree and a non-membership degree. The Intuitionistic fuzzy set widely used in application's view point in many fields, such as decision making [18, 19], medical diagnosis [9, 14] and cluster analysis[15, 16].Fuzzy Relation are most important notion of fuzzy set theory and fuzzy systems theory. Intuitionistic fuzzy relations were give many results by [3][4][10]. Xu [18] defined some new Intuitionistic preference relation and studied their properties.

The Intuitionistic fuzzy set to picture fuzzy sets were generalized by Cuong[6,7]. He proposed the notion for picture fuzzy relations and studied their operations and properties. Picture Fuzzy set give the three degrees of an element called degree of positive membership, degree of Neutral membership and degree of negative membership respectively. Picture fuzzy sets based models may be adequate in situations when we face human opinions involving more answers of types: yes, abstain, no, refusal. Bosc and Pivert[2] said that "Bipolarity refers to the propensity of the human mind to reason and make decisions on the basis of positive and negative effects. There are two information, first is the positive information that is possible, satisfactory, desired and being acceptable. On the other hand, negative information states that impossible, rejected or forbidden. Lee [11, 12] proposed the concept of bipolar fuzzy set which is a generalization of the fuzzy sets. However aggregation operator is commonly used in multi-attribute decision making problems.

In this paper, we introduce new operations on bipolar picture fuzzy and their relations. Also we develop, Bipolar Picture Fuzzy Geometric operators and discuss their properties. We further propose Bipolar Picture Fuzzy Multiple Attribute Decision making problems. A numerical example for the method is given to demonstrate the effectiveness and application of our proposed method.

II. PRELIMINARIES:

Definition 2.1: [20] Let X be the universe of discourse, then a fuzzy set is defined as: A= $\{<x, \mu_A(x)>: x\in X\}$ which is characterized by a membership function μ_A: X→ [0, 1], where μ_A denotes the degree of membership of the element *x* to the set *A*.

Definition 2.2:[1] An Intuitionistic Fuzzy Set in *X* is given by *A* = $\{<x, \mu_A(x), \gamma_A(x)>: x\in X\}$ which is characterized by a membership function *A: X*→ [0,1] and a non-membership function *A: X* → [0,1], with the condition $0\leq \mu_A(x)+ \gamma_A(x) \leq 1, \forall x\in X$ where the numbers $\mu_A(x)$ and $\gamma_A(x)$ represent the degree of membership and the degree of non-membership of the element *x* to the set *A*, respectively.

Definition 2.3: [17] Let $\alpha = (\mu_\alpha, \gamma_\alpha)$ and $\beta = (\mu_\beta, \gamma_\beta)$ be two Intuitionistic fuzzy numbers, then

(1) $\alpha . \beta = (\mu_\alpha \mu_\beta , \gamma_\alpha \gamma_\beta - \gamma_\alpha \gamma_\beta)$

(2) $\alpha^\lambda = (\mu_\alpha{}^\lambda , 1\text{-}(1\text{-} \gamma_\alpha)^\lambda), \lambda > 0$.

Definition 2.4: [8] A Picture Fuzzy set (PFS) A on a Universe X is an object in the form of

A=$\{(x, \mu_A(x), \eta_A(x), \gamma_A(x): x\in X)\}$ where $\mu_A(x),\eta_A(x),\gamma_A(x) \in [0,1]$.Here $\mu_A(x)\in[0,1]$ called positive membership, $\eta_A(x) \in[0,1]$ called degree of neutral membership and $\gamma_A(x) \in[0,1]$ called degree of negative membership. These memberships satisfying the condition $\forall x \in X$,

$\mu_A(x) + \eta_A(x) + \gamma_A(x) \leq 1$. Then, for $x \in X$, $\rho_A(x)= 1\text{-} \mu_A(x) - \eta_A(x) - \gamma_A(x)$ could be called the degree of refusal membership of in A.

Definition 2.5: [8] For every two PFSs *A* and *B*, defined some operations as following.

(1) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$, $\eta_A(x) \geq \eta_B(x)$ and $\gamma_A(x) \geq \gamma_B(x); \forall x \in X$;
(2) A = B iff $A \subseteq B$ and $B \subseteq A$
(3) AUB= {(x, max ($\mu_A(x)$, $\mu_B(x)$), min ($\eta_A(x)$, $\eta_B(x)$), and min (γA (x), γB (x)):x∈X}
(4) A∩B= {(x, min ($\mu_A(x)$, $\mu_B(x)$), max ($\eta_A(x)$, $\eta_B(x)$), and max (γA (x), γB (x)):x∈X}
(5) A^c= {(x, $\gamma_A(x)$, $\eta_A(x)$, $\mu_A(x)$):x∈X}

Several properties of these operations were also discussed:
(1) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$
(2) $\overline{\overline{A}} = A$
(3) Operations U and ∩ are commutative, associative and distributive.
(4) Operations ∩, *co* and U satisfy the De Morgan law.

Definition 2.6: [8] A picture fuzzy relation R is a picture fuzzy subset of XxY, i.e. R is given by
$R = \{((x, y), \mu_R\ (x, y), \eta_R\ (x, y), \gamma_R\ (x, y)): x \in X, y \in Y\}$ where $\forall$ (x, y) $\in X \times Y$ and also
$\mu_R, \eta_R, \gamma_R : X \times Y \rightarrow [0,1]$ and satisfying the condition $\forall$ (x, y) $\in X \times Y$, $\mu_R(x) + \eta_R(x) + \gamma_R(x) \leq 1$. We will denote with PFS ($X \times Y$) the set of all Picture fuzzy subsets in $X \times Y$.

Definition 2.7: [8] A binary Picture fuzzy relation between X and Y, we can define R^{-1} between Y and X by means of
$\mu_{R^{-1}}\ (y, x) = \mu_R\ (x, y)$, $\eta_{R^{-1}}\ (y,) = \eta_R\ (x,)$, $\gamma_{R^{-1}}\ (y,x) = \gamma_R\ (x,y)$, $\bar{\gamma}_{R^{-1}}\ (y,x) = \bar{\gamma}_R\ (x,y)$, $\forall (x, y) \in X \times Y$
to which we call inverse relation of R.

Definition 2.8: [5] Let α = {(x, $\mu_\alpha(x)$, $\eta_\alpha(x)$, $\gamma_\alpha(x)$:x∈X} and β = {(x, $\mu_\beta(x)$, $\eta_\beta(x)$, $\gamma_\beta(x)$: x∈X} be the two picture fuzzy numbers, then

a. $\alpha . \beta = (\mu_\alpha + \eta_\alpha)(\mu_\beta + \eta_\beta) - \eta_\alpha \eta_\beta, \eta_\alpha \eta_\beta, 1-(1-\gamma_\alpha)(1-\gamma_\beta)$
b. $\alpha^\lambda = (\mu_\alpha + \eta_\alpha)^\lambda(\mu_\beta + \eta_\beta)^\lambda - \eta_\alpha^\lambda \eta_\beta^\lambda, \eta_\alpha^\lambda \eta_\beta^\lambda, 1-(1-\gamma_\alpha)^\lambda(1-\gamma_\beta)^\lambda, \lambda > 0$.

Definition 2.9: [5] Let α = {(x, $\mu_\alpha(x)$, $\eta_\alpha(x)$, $\gamma_\alpha(x)$:x∈X} be the picture fuzzy numbers, then a score function S can be defined as $S(\alpha) = \mu_\alpha - \gamma_\alpha$ and the accuracy function H is given by
$H(\alpha) = \mu_\alpha + \eta_\alpha + \gamma_\alpha$ where $S(\alpha) \in [-1,1]$ and $H(\alpha) \in [0, 1]$. Then, for two picture fuzzy numbers α and β
(i) if $S(\alpha) > S(\beta)$, then α is superior to β, denoted by $\alpha \succ \beta$.
(ii) if $S(\alpha) = S(\beta)$, then
(1) $H(\alpha) = H(\beta)$, implies that α is equivalent to β, denoted by α ~β.
(2) $H(\alpha) > H(\beta)$, implies that α is superior to β, denoted by $\alpha \succ \beta$.

Definition 2.10: [13] Let X be a non-empty set. Then, a bipolar-valued fuzzy set is denoted by A_{BF}, defined by A_{BF} = {(x, $\mu_A^+(x)$, $\mu_A^-(x)$: x ∈X)} where $\mu_A^+ : X \rightarrow [0, 1]$ and $\mu_A^- : X \rightarrow [0,-1]$.The positive membership degree $\mu_A^+(x)$ denotes the satisfaction degree of an element x to the property corresponding to A_{BF} and the negative membership degree $\mu_A^-(x)$ denotes the satisfaction degree of x to some implicit counter property of A_{BF}.

III. BIPOLAR PICTURE FUZZY SET (BPFS)

Definition 3.1: A bipolar Picture Fuzzy set (BPFS) A on a Universe X is an object in the form of

$$A=\{(x, \mu_A^+(x),\eta_A^+(x),\gamma_A^+(x), \mu_A^-(x),\eta_A^-(x),\gamma_A^-(x)):x\in X\}$$

where $\mu_A^+(x),\eta_A^+(x),\gamma_A^+(x) \in [0,1]$ and $\mu_A^-(x),\eta_A^-(x),\gamma_A^-(x) \in [-1,0]$.Here $\mu_A^+(x),\mu_A^-(x)$ are positive and negative of degree of positive membership, $\eta_A^+(x),\eta_A^-(x)$ are positive and negative of degree of neutral membership. $\gamma_A^+(x),\gamma_A^-(x)$ are positive and negative of degree of negative membership.

Besides, $\mu_A^+, \mu_A^-, \eta_A^+, \eta_A^-, \gamma_A^+, \gamma_A^-$ satisfying the condition $\forall$ x $\in X$, $\mu_A^+(x) + \eta_A^+(x) + \gamma_A^+(x) \leq 1$ and $\mu_A^-(x) + \eta_A^-(x) + \gamma_A^-(x) \geq -1$. Then for every x∈X, $\rho_A^+(x) = 1 - \mu_A^+ - \eta_A^+ - \gamma_A^+$ and $\rho_A^-(x) = 1 - \mu_A^- - \eta_A^- - \gamma_A^-$ called refusal membership of x in A.

Definition 3.2: Let A and B be two bipolar picture fuzzy sets in X defined as
$A=\{(x, \mu_A^+(x),\eta_A^+(x),\gamma_A^+(x), \mu_A^-(x),\eta_A^-(x),\gamma_A^-(x)):x\in X\}$ and $B=\{(x, \mu_B^+(x),\eta_B^+(x),\gamma_B^+(x), \mu_B^-(x), \eta_B^-(x), \gamma_B^-(x)):x\in X\}$. Then the following operators are defined as

(i) Complement

A= {< (1- $\mu_A^+(x)$), (-1- $\mu_A^-(x)^-$), (1- $\eta_A^+(x)$), (-1- $\eta_A^-(x)$), (1- $\gamma_A^+(x)$), (-1- $\gamma_A^-(x)$)>}

(ii) Union of two BPFS

A U B = {(max ($\mu_A^+(x)$, $\mu_B^+(x)$), min ($\eta_A^+(x)$, $\eta_B^+(x)$), min ($\gamma_A^+(x)$, $\gamma_B^+(x)$,
max ($\mu_A^-(x)$,$\mu_B^-(x)$),min ($\eta_A^-(x)$,$\eta_B^-(x)$), min ($\gamma_A^-(x)$,$\gamma_B^-(x)$))}

(iii) Intersection of two BPFS

$A \cap B = \{(\min(\mu_A^+(x), \mu_B^+(x)), \max(\eta_A^+(x), \eta_B^+(x)), \max(\gamma_A^+(x), \gamma_B^+(x)),$
$\min(\mu_A^-(x), \mu_B^-(x)), \max(\eta_A^-(x), \eta_B^-(x)), \max(\gamma_A^-(x), \gamma_B^-(x)))\}$

(iv) A=B iff

$\mu_A^+(x) = \mu_B^+(x)$, $\eta_A^+(x) = \eta_B^+(x)$, $\gamma_A^+(x) = \gamma_B^+(x)$,
$\mu_A^-(x) = \mu_B^-(x)$, $\eta_A^-(x) = \eta_B^-(x)$, $\gamma_A^-(x) = \gamma_B^-(x)$

(v) A ⊆ B iff

$\mu_A^+(x) \le \mu_B^+(x)$, $\eta_A^+(x) \ge \eta_B^+(x)$, $\gamma_A^+(x) \ge \gamma_B^+(x)$,
$\mu_A^-(x) \le \mu_B^-(x)$, $\eta_A^-(x) \ge \eta_B^-(x)$, $\gamma_A^-(x) \ge \gamma_B^-(x)$

Definition 3.3: (i) A Bipolar Picture fuzzy set relation is defined as a Bipolar Picture fuzzy subset of $X \times Y$ having the form

$$R = \{((x, y), \mu_R^+(x, y), \mu_R^-(x, y), \eta_R^+(x, y), \eta_R^-(x, y), \gamma_R^+(x, y), \gamma_R^-(x, y)) : x \in X, y \in \}$$

where $\forall (x, y) \in X \times Y$ and also $\mu_R^+, \eta_R^+, \gamma_R^+ : X \times Y \to [0,1]$, $\mu_R^-, \eta_R^-, \gamma_R^- : X \times Y \to [-1,0]$. We will denote with BPFS $(X \times Y)$ the set of all bipolar Picture fuzzy subsets in $X \times Y$.

(ii) A binary bipolar Picture fuzzy relation between X and Y, we can define R^{-1} between Y and X by means of $\mu_{R^{-1}}^+(y, x) = \mu_R^+(x, y)$, $\mu_{R^{-1}}^-(y, x) = \mu_R^-(x, y)$, $\eta_{R^{-1}}^+(y,x) = \eta_R^+(x,y)$, $\eta_{R^{-1}}^-(y,x) = \eta_R^-(x,y)$, $\gamma_{R^{-1}}^+(y,x) = \gamma_R^+(x,y)$, $\gamma_{R^{-1}}^-(y,x) = \gamma_R^-(x,y)$ $\forall (x, y) \in X \times Y$ to which we call inverse relation of R.

(iii) Let R and P be two bipolar picture fuzzy relations between X and Y, For every $(x, y) \in X \times Y$ .We can define,

a. $R \le P \Leftrightarrow \{ (x, y), \mu_R^+(x, y) \le \mu_P^+(x, y), \mu_R^-(x, y) \le \mu_P^-((x, y), \eta_R^+(x, y) \ge \eta_P^+(x, y),$
$\eta_R^-(x, y) \ge \eta_P^-(x, y), \gamma_R^+(x, y) \ge \gamma_P^+(x, y), \gamma_R^-(x, y) \ge \gamma_P^-(x, y)\}$

b. $R^c = \{(x, y), (1- \mu_R^+(x, y)), (-1- \mu_R^-(x, y)), (1- \eta_R^+(x, y)), (-1- \eta_R^-(x, y)),$
$(1- \gamma_R^+(x, y)),(-1- \gamma_R^-(x, y))\}$

c. $R \vee P = \{(x, y), \mu_R^+(x, y) \vee \mu_P^+(x, y), \mu_R^-(x, y) \vee \mu_P^-(x, y), \eta_R^+(x, y) \wedge \eta_P^+(x, y),$
$\eta_R^-(x, y) \wedge \eta_P^-(x, y), \gamma_R^+(x, y) \wedge \gamma_P^+(x, y), \gamma_R^-(x, y) \wedge \gamma_P^-(x, y)\}$

d. $R \wedge P = \{(x, y), \mu_R^+(x, y) \wedge \mu_P^+(x, y), \mu_R^-(x, y) \wedge \mu_P^-(x, y), \eta_R^+(x, y) \vee \eta_P^+(x, y),$
$\eta_R^-(x, y) \vee \eta_P^-(x, y), \gamma_R^+(x, y) \vee \gamma_P^+(x, y), \gamma_R^-(x, y) \vee \gamma_P^-(x, y)\}$

Theorem 3.4: Let R, P, Q be three elements of bipolar picture fuzzy relations (X×Y)
(i) $R \le P \Rightarrow R^{-1} \le P^{-1}$ (ii) $(R \vee P)^{-1} = R^{-1} \vee P^{-1}$ (iii) $(R \wedge P)^{-1} = R^{-1} \wedge P^{-1}$ (iv) $(R^{-1})^{-1} = R$
(v) $R \wedge (P \vee Q) = (R \wedge P) \vee (R \wedge Q)$ and $R \vee (P \wedge Q) = (R \vee P) \wedge (R \vee Q)$
(vi) $R \vee P \ge R$, $R \vee P \ge P$, $R \wedge P \le R$, $R \wedge P \le P$
(vii) If $R \ge P$ and $R \ge Q$; then $R \ge (P \vee Q)$ (viii) If $R \le P$ and $R \le Q$; then $R \le (P \vee Q)$

Proof:

(i) If $R \le P$, then for every $(x, y) \in X \times Y$

$\mu_{R^{-1}}^+(y,x) = \mu_R^+(x, y) \le \mu_P^+(x, y) = \mu_{P^{-1}}^+(y,x)$ $\mu_{R^{-1}}^-(y, x) = \mu_R^-(x, y) \le \mu_P^-(x, y) = \mu_{P^{-1}}^-(y,x)$

$\eta_{R^{-1}}^+(y,x) = \eta_R^+(x,y) \ge \eta_P^+(x,y) = \eta_{P^{-1}}^+(y,x)$ $\eta_{R^{-1}}^-(y,x) = \eta_R^-(x,y) \ge \eta_P^-(x,y) = \eta_{P^{-1}}^-(y,x)$

$\gamma_{R^{-1}}^+(y,x) = \gamma_R^+(x,y) \ge \gamma_P^+(x,y) = \gamma_{P^{-1}}^+(y,x)$ $\gamma_{R^{-1}}^-(y,x) = \gamma_R^-(x,y) \le \gamma_P^-(x,y) = \gamma_{P^{-1}}^-(y,x)$

Hence $R^{-1} \le P^{-1}$.

(ii) $(R \vee P)^{-1} = R^{-1} \vee P^{-1}$

$\mu_{(R\vee P)^{-1}}^+(y,x) = \mu_{(R\vee P)}^+(x,y) = \mu_R^+(x,y) \vee \mu_P^+(x,y) = \mu_{R^{-1}}^+(y,x) \vee \mu_{P^{-1}}^+(y,x) = \mu_{R^{-1}\vee P^{-1}}^+(y,x)$

$\mu_{(R\vee P)^{-1}}^-(y,x) = \mu_{(R\vee P)}^-(x,y) = \mu_R^-(x,y) \vee \mu_P^-(x,y) = \mu_{R^{-1}}^-(y,x) \vee \mu_{P^{-1}}^-(y,x) = \mu_{R^{-1}\vee P^{-1}}^-(y,x)$

The proof is similar for $\eta_{(R\vee P)^{-1}}^+(y,x) = \eta_{R^{-1}\vee P^{-1}}^+(y,x)$, $\eta_{(R\vee P)^{-1}}^-(y,x) = \eta_{R^{-1}\vee P^{-1}}^-(y,x)$ and $\gamma_{(R\vee P)^{-1}}^+(y,x) = \gamma_{R^{-1}\vee P^{-1}}^+(y,x)$, $\gamma_{(R\vee P)^{-1}}^-(y,x) = \gamma_{R^{-1}\vee P^{-1}}^-(y,x)$.

(iii) $(R \wedge P)^{-1} = R^{-1} \wedge P^{-1}$: The proof is similar to (ii).

(iv) $(R^{-1})^{-1} = R$: The proof follows from the definition.

(v) $R \wedge (P \vee Q) = (R \wedge P) \vee (R \wedge Q)$ and $R \vee (P \wedge Q) = (R \vee P) \wedge (R \vee Q)$

Proof:

$\mu^{+}_{(R\wedge(P\vee Q))}(x,y) = \mu^{+}_{R}(x,y) \wedge \{\mu^{+}_{P}(x,y) \vee \mu^{+}_{Q}(x,y)\} = \mu^{+}_{R}(x,y) \wedge \mu^{+}_{P}(x,y) \vee \mu^{+}_{R}(x,y) \wedge \mu^{+}_{Q}(x,y) = \mu^{+}_{R\wedge P}(x,y) \vee \mu^{+}_{R\wedge Q}(x,y) = \mu^{+}_{(R\wedge P)\vee(R\wedge Q)}(x,y)$

Similarly for $\mu^{-}_{(R\wedge(P\vee Q))}(x,y)$, $\eta^{+}_{(R\wedge(P\vee Q))}(x,y)$, $\eta^{-}_{(R\wedge(P\vee Q))}(x,y)$, $\gamma^{+}_{(R\wedge(P\vee Q))}(x,y)$, $\gamma^{-}_{(R\wedge(P\vee Q))}(x,y)$.

Hence $R \wedge (P \vee Q) = (R \wedge P) \vee (R \wedge Q)$.

The proof is analogous to the above one, in the case of $R \vee (P \wedge Q) = (R \vee P) \wedge (R \vee Q)$.

(vi) $R \vee P \geq R$, $R \vee P \geq P$, $R \wedge P \leq R$, $R \wedge P \leq P$: The proof is obvious.

(vii) If $R \geq P$ and $R \geq Q$; then $R \geq (P \vee Q)$

Proof: If $R \geq P$ and $R \geq Q$, $\mu^{+}_{R}(x,y) \geq \mu^{+}_{P}(x,y)$, $\mu^{-}_{R}(x,y) \geq \mu^{-}_{P}(x,y)$, $\eta^{+}_{R}(x,y) \leq \eta^{+}_{P}(x,y)$, $\eta^{-}_{R}(x,y) \leq \eta^{-}_{P}(x,y)$, $\gamma^{+}_{R}(x,y) \leq \gamma^{+}_{P}(x,y)$, $\gamma^{-}_{R}(x,y) \leq \gamma^{-}_{P}(x,y)$, $\mu^{+}_{R}(x,y) \geq \mu^{+}_{Q}(x,y)$, $\mu^{-}_{R}(x,y) \geq \mu^{-}_{Q}(x,y)$, $\eta^{+}_{R}(x,y) \leq \eta^{+}_{Q}(x,y)$, $\eta^{-}_{R}(x,y) \leq \eta^{-}_{Q}(x,y)$, $\gamma^{+}_{R}(x,y) \leq \gamma^{+}_{Q}(x,y)$, $\gamma^{-}_{R}(x,y) \leq \gamma^{-}_{Q}(x,y)$

$\mu^{+}_{R}(x,y) \geq \mu^{+}_{P}(x,y) \vee \mu^{+}_{Q}(x,y) \Rightarrow \mu^{+}_{R}(x,y) \geq \mu^{+}_{P\vee Q}(x,y) \Rightarrow R \geq (P \vee Q)$, similarly for $\mu^{-}_{R}(x,y) \geq \mu^{-}_{P\vee Q}(x,y)$, $\eta^{+}_{R}(x,y) \leq \eta^{+}_{P\vee Q}(x,y)$, $\eta^{-}_{R}(x,y) \leq \eta^{-}_{P\vee Q}(x,y)$, $\gamma^{+}_{R}(x,y) \leq \gamma^{+}_{P\vee Q}(x,y)$, $\gamma^{-}_{R}(x,y) \leq \gamma^{-}_{P\vee Q}(x,y)$. Therefore $R \geq (P \vee Q)$.

(viii) If $R \leq P$ and $R \leq Q$; then $R \leq (P \vee Q)$: The proof is similar as (vii).

Definition 3.5: Let $\alpha = \{(x, \mu^{+}_{\alpha}(x), \eta^{+}_{\alpha}(x), \gamma^{+}_{\alpha}(x), \mu^{-}_{\alpha}(x), \eta^{-}_{\alpha}(x), \gamma^{-}_{\alpha}(x)) : x \in X\}$ and $\beta = \{(x, \mu^{+}_{\beta}(x), \eta^{+}_{\beta}(x), \gamma^{+}_{\beta}(x), \mu^{-}_{\beta}(x), \eta^{-}_{\beta}(x), \gamma^{-}_{\beta}(x)) : x \in X\}$ be the two bipolar picture fuzzy numbers, then

a. $\alpha.\beta = (\mu^{+}_{\alpha} + \eta^{+}_{\alpha})(\mu^{+}_{\beta} + \eta^{+}_{\beta}) - \eta^{+}_{\alpha}\eta^{+}_{\beta}$, $\eta^{+}_{\alpha}\eta^{+}_{\beta}$, $1-(1-\gamma^{+}_{\alpha})(1-\gamma^{+}_{\beta})$,

$-(((-\mu^{-}_{\alpha})+(-\eta^{-}_{\alpha}))((-\mu^{-}_{\beta})+(-\eta^{-}_{\beta})) - (-\eta^{-}_{\alpha})(-\eta^{-}_{\beta}))$, $-((-\eta^{-}_{\alpha})(-\eta^{-}_{\beta}))$, $1-(1-\gamma^{-}_{\alpha})(1-\gamma^{-}_{\beta})$

b. $\alpha^{\lambda} = (\mu^{+}_{\alpha} + \eta^{+}_{\alpha})^{\lambda} - (\eta^{+}_{\alpha})^{\lambda}$, $(\eta^{+}_{\alpha})^{\lambda}$, $1-(1-\gamma^{+}_{\alpha})^{\lambda}$, $-(((-\mu^{-}_{\alpha})+(-\eta^{-}_{\alpha}))^{\lambda} - (-\eta^{-}_{\alpha})^{\lambda}$, $-(-\eta^{-}_{\alpha})^{\lambda}$,

$1-(1-\gamma^{-}_{\alpha})^{\lambda}$ $(\lambda>0)$

Definition 3.6: Let $\alpha = \{(x, \mu^{+}_{\alpha}(x), \eta^{+}_{\alpha}(x), \gamma^{+}_{\alpha}(x), \mu^{-}_{\alpha}(x), \eta^{-}_{\alpha}(x), \gamma^{-}_{\alpha}(x)) : x \in X\}$ be bipolar picture fuzzy numbers, then the

Score function $S(\alpha) = (\mu^{+}_{\alpha} + 1 - \eta^{+}_{\alpha} + 1 - \gamma^{+}_{\alpha} + 1 + \mu^{-}_{\alpha} - \eta^{-}_{\alpha} - \gamma^{-}_{\alpha})/6$

Accuracy function $H(\alpha) = \mu^{+}_{\alpha} - \gamma^{+}_{\alpha} + \mu^{-}_{\alpha} - \gamma^{-}_{\alpha}$

(i) If $S(\alpha) > S(\beta)$, then α is superior to β, $\alpha > \beta$.

(ii) If $S(\alpha) = S(\beta)$, then (a). $H(\alpha) = H(\beta)$ implies that α is equivalent to β, $\alpha \sim \beta$.
(b). $H(\alpha) > H(\beta)$ implies that α is superior to β, $\alpha > \beta$.

IV. BIPOLAR PICTURE FUZZY GEOMETRIC OPERATOR:

Definition 4.1: Let A_j (j=1, 2… n) is the collect ion of BPFNS, we define bipolar picture fuzzy weighted geometric operator (BPFWG),

$BPFWG_{\omega}(A_1, A_2, \ldots, A_n) = \prod_{j=1}^{n} A_j^{\omega_j}$ where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weighted vector of A_j (j=1,2,…,n) and $\omega_j > 0$, $\sum_{j=1}^{n} \omega_j = 1$.

Theorem 4.2: Let A_j (j=1, 2… n) is the collect ion of BPFNS, then their aggregated value by the BPFWG operator is also a BPFNS,

$$BPFWG_{\omega}(A_1, A_2 \ldots A_n) = \Big(\prod_{i=1}^{n} (\mu^{+}_{i} + \eta^{+}_{i})^{\omega_i} - \prod_{i-1}^{n} (\eta^{+}_{i})^{\omega_i}, \prod_{i=1}^{n} (\eta^{+}_{i})^{\omega_i}, 1-\prod_{i=1}^{n} (1-\gamma^{+}_{i})^{\omega_i},$$

$$-\Big(\prod_{i=1}^{n} ((-\mu^{-}_{i}) + (-\eta^{-}_{i}))^{\omega_i} - \prod_{i=1}^{n} (-\eta^{-}_{i})^{\omega_i}\Big), -\prod_{i=1}^{n} (-\eta^{-}_{i})^{\omega_i}, 1-\prod_{i=1}^{n} (1-\gamma^{-}_{i})^{\omega_i}\Big)$$

Proof: We prove that the theorem, by using mathematical induction on n,
If for n=2, we get

$A_1^{\omega_1} = ((\mu^{+}_{1} + \eta^{+}_{1})^{\omega_1} - (\eta^{+}_{1})^{\omega_1}, (\eta^{+}_{1})^{\omega_1}, 1-(1-\gamma^{+}_{1})^{\omega_1}, -(((-\mu^{-}_{1}) + (-\eta^{-}_{1}))^{\omega_1} - (-\eta^{-}_{1})^{\omega_1}), -(-\eta^{-}_{1})^{\omega_1},$
$1-(1-\gamma^{-}_{1})^{\omega_1})$

$A_2^{\omega_2} = ((\mu^{+}_{2} + \eta^{+}_{2})^{\omega_2} - (\eta^{+}_{2})^{\omega_1}, (\eta^{+}_{2})^{\omega_2}, 1-(1-\gamma^{+}_{2})^{\omega_2}, -(((-\mu^{-}_{2}) + (-\eta^{-}_{2}))^{\omega_2} - (-\eta^{-}_{2})^{\omega_1}), -(-\eta^{-}_{2})^{\omega_2},$
$1-(1-\gamma^{-}_{2})^{\omega_2})$

Then,

$A_1^{\omega_1} . A_2^{\omega_2} = ((\mu^{+}_{1} + \eta^{+}_{1})^{\omega_1}(\mu^{+}_{2} + \eta^{+}_{2})^{\omega_2} - (\eta^{+}_{1})^{\omega_1}(\eta^{+}_{2})^{\omega_1}, (\eta^{+}_{1})^{\omega_1}(\eta^{+}_{2})^{\omega_1}, 1-(1-\gamma^{+}_{1})^{\omega_1}(1-\gamma^{+}_{2})^{\omega_2},$

$- (((-\mu_1^-) + (-\eta_1^-))^{\omega_1} ((-\mu_2^-) + (-\eta_2^-))^{\omega_2} - (-\eta_1^-)^{\omega_1} (-\eta_2^-)^{\omega_2}), (-\eta_1^-)^{\omega_1} (-\eta_2^-)^{\omega_2}, 1-(1-\gamma_1^-)^{\omega_1} (1-\gamma_2^-)^{\omega_2})$

Thus the theorem holds. If for n= k then,

$$BPFWG_\omega (A_1,\ldots A_k) = \Big(\prod_{i=1}^{k} (\mu_i^+ + \eta_i^+)^{\omega_i} - \prod_{i=1}^{k} (\eta_i^+)^{\omega_i}, \prod_{i=1}^{k} (\eta_i^+)^{\omega_i}, 1-\prod_{i=1}^{k} (1-\gamma_i^+)^{\omega_i},$$

$$-\Big(\prod_{i=1}^{k} ((-\mu_i^-) + (-\eta_i^-))^{\omega_i} - \prod_{i=1}^{k} (-\eta_i^-)^{\omega_i}\Big), -\prod_{i=1}^{k} (-\eta_i^-)^{\omega_i}, 1-\prod_{i=1}^{k} (1-\gamma_i^-)^{\omega_i}\Big)$$

When n= k+1, we have

$$\prod_{i=1}^{k+1} A_i^{\omega_i} = \prod_{i=1}^{k} A_i^{\omega_i} A_{k+1}^{\omega_{k+1}}$$

$$= \Big(\prod_{i=1}^{k} (\mu_i^+ + \eta_i^+)^{\omega_i} - \prod_{i=1}^{k} (\eta_i^+)^{\omega_i}, \prod_{i=1}^{k} (\eta_i^+)^{\omega_i}, 1-\prod_{i=1}^{k} (1-\gamma_i^+)^{\omega_i},$$

$$-\Big(\prod_{i=1}^{k} ((-\mu_i^-) + (-\eta_i^-))^{\omega_i} - \prod_{i=1}^{k} (-\eta_i^-)^{\omega_i}\Big), -\prod_{i=1}^{k} (-\eta_i^-)^{\omega_i}, 1-\prod_{i=1}^{k} (1-\gamma_i^-)^{\omega_i}\Big).$$

$$(\mu_{k+1}^+ + \eta_{k+1}^+)^{\omega_{k+1}} - (\eta_{k+1}^+)^{\omega_{k+1}}, (\eta_{k+1}^+)^{\omega_{k+1}}, 1-(1-\gamma_{k+1}^+)^{\omega_{k+1}}, -(((-\mu_{k+1}^-) + (-\eta_{k+1}^-))^{\omega_{k+1}} - (-\eta_{k+1}^-)^{\omega_{k+1}}),$$

$$-(-\eta_{k+1}^-)^{\omega_{k+1}}, 1-(1-\gamma_{k+1}^-)^{\omega_{k+1}})$$

$$= \Big(\prod_{i=1}^{k+1} (\mu_i^+ + \eta_i^+)^{\omega_i} - \prod_{i=1}^{k+1} (\eta_i^+)^{\omega_i}, \prod_{i=1}^{k+1} (\eta_i^+)^{\omega_i}, 1-\prod_{i=1}^{k+1} (1-\gamma_i^+)^{\omega_i},$$

$$-\Big(\prod_{i=1}^{k+1} ((-\mu_i^-) + (-\eta_i^-))^{\omega_i} - \prod_{i=1}^{k+1} (-\eta_i^-)^{\omega_i}\Big), -\prod_{i=1}^{k+1} (-\eta_i^-)^{\omega_i}, 1-\prod_{i=1}^{k+1} (1-\gamma_i^-)^{\omega_i}\Big)$$

(i.e) The theorem holds for n= k+1, by the principle of mathematical induction, the theorem holds for all n.

$$\Big(\prod_{i=1}^{n} (\mu_i^+ + \eta_i^+)^{\omega_i} - \prod_{i=1}^{n} (\eta_i^+)^{\omega_i}, \prod_{i=1}^{n} (\eta_i^+)^{\omega_i}, 1-\prod_{i=1}^{n} (1-\gamma_i^+)^{\omega_i},$$

$$-\Big(\prod_{i=1}^{n} ((-\mu_i^-) + (-\eta_i^-))^{\omega_i} - \prod_{i=1}^{n} (-\eta_i^-)^{\omega_i}\Big), -\prod_{i=1}^{n} (-\eta_i^-)^{\omega_i}, 1-\prod_{i=1}^{n} (1-\gamma_i^-)^{\omega_i}\Big) \in [0,1]x[0,-1].$$

∴ The Result of $BPFWG_\omega$ (A_1, A_2,.. A_n) is also a BPFN.

When we need to weight the ordered positions of the bipolar picture fuzzy arguments instead of weighting the argument, BPWG can be generalized to BPFOWG

Definition 4.3: Let A_j (j=1, 2… n) is the collect ion of BPFNS, we define bipolar picture fuzzy ordered weighted geometric operator (BPFOWG),

$BPFOWG_\omega (A_1,A_2,\ldots.A_n) = \prod_{j=1}^{n} A_{\sigma(j)}^{\omega_j}$ where $\omega=(\omega_1,\omega_2,\ldots..,\omega_n)^T$ be the weighted vector of A_j (j=1,2,…,n) and $\omega_j > 0$,

$$\sum_{j=1}^{n} \omega_j = 1.$$

Theorem 4.4: Let A_j (j=1, 2… n) is the collect ion of BPFNS, then their aggregated value by the BPFWG operator is also a BPFNS, $BPFOWG_\omega$ (A_1, A2… An) =

$$\Big(\prod_{i=1}^{n} (\mu_{\sigma(i)}^+ + \eta_{\sigma(i)}^+)^{\omega_i} - \prod_{i=1}^{n} (\eta_{\sigma(i)}^+)^{\omega_i}, \prod_{i=1}^{n} (\eta_{\sigma(i)}^+)^{\omega_i}, 1-\prod_{i=1}^{n} (1-\gamma_{\sigma(i)}^+)^{\omega_i},$$

$$-\Big(\prod_{i=1}^{n} ((-\mu_{\sigma(i)}^-) + (-\eta_{\sigma(i)}^-))^{\omega_i} - \prod_{i=1}^{n} (-\eta_{\sigma(i)}^-)^{\omega_i}\Big), -\prod_{i=1}^{n} (-\eta_{\sigma(i)}^-)^{\omega_i}, 1-\prod_{i=1}^{n} (1-\gamma_{\sigma(i)}^-)^{\omega_i}\Big)$$

The proof is similar to the above theorem 4.2.

Definition 4.5: Let A_j (j=1, 2… n) is the collect ion of BPFNS, we define bipolar picture fuzzy hybrid weighted geometric operator (BPFHWG),

$BPFHWG_\omega (A_1,A_2,\ldots.A_n) = \prod_{j=1}^{n} \tilde{A}_{\sigma(j)}^{\omega_j}$ where $\omega=(\omega_1,\omega_2,\ldots..,\omega_n)^T$ be the weighted vector of A_j (j=1,2,…,n) and $\omega_j > 0$,

$\sum_{j=1}^{n}\omega_j$ =1 and $A_{\sigma(j)}$ is the j[th] largest element of the bipolar picture fuzzy arguments ($\tilde{A}_j = A_j^{n\omega_j}$), ω_j is the weighting vector of BPF argument A_j with $\omega_j > 0$, $\sum_{j=1}^{n}\omega_j$ =1 and n is the balancing coefficient.

Theorem 4.6: Let A_j (j=1, 2… n) is the collect ion of BPFNS, then their aggregated using value by the BPFWG operator is also a BPFNS,

BPFHWG$_\omega$ ($A_1,A_2,\ldots A_n$) =

$$\Big(\prod_{i=1}^{n}(\tilde{\mu}^{+}_{\sigma(i)}+\tilde{\eta}^{+}_{\sigma(i)})^{\omega_i}-\prod_{i=1}^{n}(\tilde{\eta}^{+}_{\sigma(i)})^{\omega_i},\ \prod_{i=1}^{n}(\tilde{\eta}^{+}_{\sigma(i)})^{\omega_i},\ 1-\prod_{i=1}^{n}(1-\tilde{\gamma}^{+}_{\sigma(i)})^{\omega_i},$$

$$-\big(\prod_{i=1}^{n}((-\tilde{\mu}^{-}_{\sigma(i)})+(-\tilde{\eta}^{-}_{\sigma(i)}))^{\omega_i}-\prod_{i=1}^{n}(-\tilde{\eta}^{-}_{\sigma(i)})^{\omega_i}\big),\ -\prod_{i=1}^{n}(-\tilde{\eta}^{-}_{\sigma(i)})^{\omega_i},\ 1-\prod_{i=1}^{n}(1-\tilde{\gamma}^{-}_{\sigma(i)})^{\omega_i}\Big)$$

The proof is similar to the above theorem 4.2.

IV. MULTIPLE ATTRIBUTE DECISION MAKING WITH BIPOLAR PICTURE FUZZY INFORMATION:

To represent the MADM problems for evaluation of alternatives the following notations are used. Let set of m alternatives be $F = \{F_1, F_2\ldots F_m\}$ and $G = \{G_1, G_2\ldots G_n\}$ be a set of n attributes. Values for the alternative F_i under the attribute G_j with anonymity were provided to the decision makers, considering these values as a bipolar picture fuzzy element A_{ij}.

Suppose the bipolar picture fuzzy decision matrix is the decision matrix $A = (A_{ij})_{mxn}$ where A_{ij} (i = 1, 2… m; j = 1, 2…n) are in the form of BPFNs. Evaluation of alternatives with bipolar picture fuzzy information, we apply the BPFWG operator to the MADM problems.

STEP 1: Utilize the decision information given in matrix A, and the BPFWG operator

$\tilde{A}_i$ = BPFWG$_\omega$ ($A_1,A_2,\ldots,A_n$) = $\Big(\prod_{i=1}^{n}(\mu_i^{+}+\eta_i^{+})^{\omega_i}-\prod_{i=1}^{n}(\eta_i^{+})^{\omega_i},\ \prod_{i=1}^{n}(\eta_i^{+})^{\omega_i},\ 1-\prod_{i=1}^{n}(1-\gamma_i^{+})^{\omega_i},\ -\big(\prod_{i=1}^{n}((-\mu_i^{-})+(-\eta_i^{-}))^{\omega_i}-\prod_{i=1}^{n}(-\eta_i^{-})^{\omega_i}\big),\ -\prod_{i=1}^{n}(-\eta_i^{-})^{\omega_i},\ 1-\prod_{i=1}^{n}(1-\gamma_i^{-})^{\omega_i}\Big)$ to derive the overall preference values $\tilde{A}_i$ (i = 1, 2 …, m) of the alternative F_i.

STEP 2: Calculate the scores $S(\tilde{A}_i)$ (i = 1, 2…m) of the bipolar picture fuzzy values $\tilde{A}_i$.

STEP 3: Rank the alternatives F_i (i = 1, 2… m) in with the values of $S(\tilde{A}_i)$ (i =1, 2… m) and choose the best one(s).

4.1 APPLICATION:

A customer who indent to buy a mobile phone. Suppose there are four mobile phones F_i(1,2,3,4) and we want to select the best one. Four attributes are selected by experts G_j(j=1,2,3,4) G_1:Internal storage , G_2:Processor , G_3:Camera features , G_4= Battery power. In order ,the experts are required to evaluate the four mobile phones F_i(i=1,2,3,4) under the above four attribute in anonymity. The decision matrix $A=(A_{ij})_{mxn}$ is presented in the table , A_{ij}(i=1,2,3,4 and j=1,2,3,4) are in the form of BPFNS.

STEP 1: The decision matrix provided by the patient is constructed as below:

Table 4.1: The decision matrix table

F_i /G_i	G_1	G_2	G_3	G_4
F_1	(0.1,0.2,0.3, -0.1,-0.5,-0.3)	(0.7,0.1,0.1, -0.1,-0.2,-0.3)	(0.7,0.1,0.1, -0.8,-0.1,-0.1)	(0.2,0.3,0.3, -0.2,-0.4,-0.1)
F_2	(0.2,0.4,0.3, -0.1,-0.7,-0.1)	(0.1,0.6,0.1, -0.3,-0.2,-0.1)	(0.1,0.1,0.1, -0.2,-0.3,-0.4)	(0.1,0.2,0.3, -0.4,-0.5,-0.1)
F_3	(0.1,0.5,0.4, -0.2,0.4,-0.3)	(0.2,0.4,0.5, -0.5,-0.1,-0.2)	(0.1,0.4,0.5, -0.1,-0.2,-0.3)	(0.3,0.1,0.3, -0.4,-0.3,-0.2)
F_4	(0.3,0.4,0.2, -0.2,-0.3,-0.4)	(0.3,0.4,0.2, -0.5,-0.1,-0.3)	(0.2,0.4,0.3, -0.1,-0.2,-0.4)	(0.0,0.1,0.2, -0.3,-0.4,-0.1)

The weight vector of the attributes G_j (j=1, 2, 3, 4) is ω= (0.1, 0.2, 0.3, 0.4)

We utilize the decision information given in matrix A, $\tilde{A}_i$ = BPFWG$_\omega$ ($A_1,A_2,\ldots,A_n$),

We have

$\tilde{A}_1$ =(0.4590,0.1414,0.2060,-0.4508,-0.1390,-0.1565)

$\tilde{A}_2$ =(0.1203,0.2168,0.2062,-0.2935,-0.3693,-0.1825)

$\tilde{A}_3$ =(0.2380,0.2349,0.7054,-0.2988,-0.2194,-0.2390)

$\tilde{A}_4$ =(0.0772,0.2297,0.2314,-0.2969,-0.2392,-0.2525)

STEP 2: Calculate the scores function S ($\tilde{A}_i$) (i=1, 2, 3, 4) of the overall bipolar picture fuzzy values $\tilde{A}_i$.

S ($\tilde{A}_1$) = 0.4927 S ($\tilde{A}_2$) = 0.4926 S ($\tilde{A}_3$) = 0.4113 S ($\tilde{A}_4$) = 0.4719

STEP 3: rank all the alternatives G_i(i=1,2,3,4) with the values of S($\tilde{A}_i$)(i=1,2,3,4)

$$\tilde{A}_1 > \tilde{A}_2 > \tilde{A}_4 > \tilde{A}_3 .$$

Therefore the best one is $\tilde{A}_1$ .note that > means "preferred to".

V. CONCLUSION:

In this paper, bipolar picture fuzzy sets were developed and also discussed bipolar picture fuzzy relations .Then we characterized some of its properties. The operators G_ω was developed along with the score function, accuracy function .To evaluate the values of alternatives on the attributes taken bipolar picture fuzzy decision making approach was proposed. To aggregate the bipolar picture fuzzy information to each alternative to obtain the collective values of the alternatives these operators are utilized. According to the values of the score, accuracy functions of the alternatives are ranked the most desirable one was selected. The effectiveness and application of the developed method, a numerical example was given to demonstrate.

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