TOPSIS for Solving Multi-Criteria Decision Making Problems under Bipolar Pythagorean fuzzy Information

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ABSTRACT

The paper investigates a technique for order preference by similarity to ideal solution (TOPSIS) method to solve multi-criteria decision making problems with bipolar Pythagorean fuzzy information. We define distance function to determine the distance between bipolar Pythagorean fuzzy numbers. In the decision making situation, the rating of performance values of the alternatives with respect to the criteria are provided by the decision maker in terms of bipolar Pythagorean fuzzy numbers. Also we develop bipolar Pythagorean fuzzy relative positive ideal solution (BNRPIS) and bipolar Pythagorean fuzzy relative negative ideal solution (BNRNIS). Then, the ranking order of the alternatives is obtained by TOPSIS method and most desirable alternative is selected. Finally, a numerical example is solved to demonstrate the applicability and effectiveness of the proposed approach.

Keywords: Pythagorean fuzzy sets, bipolar Pythagorean fuzzy sets, TOPSIS, multi-criteria decision making.

1. INTRODUCTION

Zadeh [22] introduced the concept of fuzzy set to deal with problems with imprecise information in 1965. However, Zadeh [22] considers one single value to express the grade of membership of the fuzzy set defined in a universe. But, it is not always possible to represent the grade of membership value by a single point. In order to overcome the difficulty, Turksen [12] incorporated interval valued fuzzy sets. In 1986, Atanassov [1] extended the concept of fuzzy sets [22] and defined intuitionistic fuzzy sets which are characterized by grade of membership and non-membership functions. Later, Lee [9,10] introduced the notion of bipolar fuzzy sets by extending the concept of fuzzy sets where the degree of membership is expanded from

[0, 1] to [-1, 1]. In a bipolar fuzzy set, if the degree of membership is zero then we say the element is unrelated to the corresponding property, the membership degree (0, 1] of an element specifies that the element somewhat satisfies the property, and the membership degree [−1, 0) of an element implies that the element somewhat satisfies the implicit counter-property [6].

Decision making is a universal process in the life of human beings, which can be described as the final outcome of some mental and reasoning processes that lead to the selection of the best alternative. In many situations, it is difficult for decision makers to precisely express a preference regarding relevant alternatives un- der several criteria, especially when relying on inaccurate, uncertain, or incomplete information. To this end, the Zadeh to address multiple criteria decision making (MCDM) problems within un- certainty. The fuzzy number is usually used by the decision maker to express his/her preference of an alternative with respect to a criterion, which means the degree to which the alternative satisfies the criterion. Afterwards, Atanassov showed that in several MCDM problems the decision makers may not only provide the degree to which the alternative satisfies the criterion but also give the degree to which the alternative dissatisfies the criterion, which is characterized by a membership degree and a non-membership degree is equal to or less than 1.IFSs have been broadly applied in real-life MCDM or multiple criteria group decision making (MCGDM) problems.

However, in several complex MCDM problems the decision makers may express their preferences of an alternative with a criterion satisfying the condition that the sum of the degree to which alternative satisfies the criterion and the degree to which the alternative dissatisfies the criterion is bigger than 1. Obviously, this situation cannot be described by using the IFS but can be described by using PFS because , the sum of membership degree and non-membership degree to which an alternative satisfying an attribute provided by decision maker(DM) may be bigger than 1, but their square sum is less than or equal to1.Yager [19] familiarized the model of Pythagorean fuzzy set. The most important and central research topic is aggregation operators. There are many scholars worked in this area and introduced several operators.

In this paper, we define distance function between two bipolar Pythagorean fuzzy sets (BPFSs) and its properties. We develop a new TOPSIS based method for solving multi-criteria group decision making problem (MCGDM) under bipolar Pythagorean fuzzy assessments and finally, give a numerical example.

2. Pythagorean Fuzzy sets

Definition 2.1[13] Let a set X be a universe of discourse. A PFS P is an object having the form:

$$P = \{\langle x, \mu_P(x), v_P(x)\rangle | x \in X\}$$

where the function $\mu_P: X \to [0,1]$ defines the degree of membership and $v_P: X \to [0,1]$ defines the degree of nonmembership of the elements $x \in X$ to P,respectively,and for every $x \in X$,it holds that

$$\left(\mu_P(x)\right)^2 + \left(v_P(x)\right)^2 \leq 1$$

For any PFS P and $x \in X$, $\pi_P(x) = \sqrt{1 - \left(\mu_P(x)\right)^2 - \left(v_P(x)\right)^2}$ is called the degree of indeterminacy of x to P.

Definition 2.2[13] Let $P_1 = (\mu_{P_1}, v_{P_1})$, $P_2 = (\mu_{P_2}, v_{P_2})$ and $P = (\mu, v)$ be three PFSs.For any $\lambda(\lambda > 0)$ represents a scalar mathematical operator,four basic operations on them are defined as follows:
Then

1) $P^C = \{\langle x, v_P(x), \mu_P(x)\rangle : x \in X\}$
2) $P_1 \cup P_2 = \left\{\langle x, \max\left(\mu_{P_1}(x), \mu_{P_2}(x)\right), \min(v_{P_1}(x), v_{P_2}(x))\rangle : x \in X\right\}$
3) $P_1 \cap P_2 = \left\{\langle x, \min\left(\mu_{P_1}(x), \mu_{P_2}(x)\right), \max(v_{P_1}(x), v_{P_2}(x))\rangle : x \in X\right\}$

Definition 2.3[13]

Let $P = (\mu_P, v_P)$, $P_1 = (\mu_{P_1}, v_{P_1})$, and $P_2 = (\mu_{P_2}, v_{P_2})$, be three PFNs and $\lambda > 0$, then their operations are defined as follows:

1) $P_1 \oplus P_2 = \left(\sqrt{\mu_{P_1}^2 + \mu_{P_2}^2 - \mu_{P_1}^2 \mu_{P_2}^2}, v_{P_1} v_{P_2}\right)$
2) $P_1 \otimes P_2 = \left(\mu_{P_1} \mu_{P_2}, \sqrt{v_{P_1}^2 + v_{P_2}^2 - v_{P_1}^2 v_{P_2}^2}\right)$
3) $\lambda P = \left(\sqrt{1 - (1 - \mu_P^2)^\lambda}, v_P^\lambda\right)$
4) $P^\lambda = \left(\mu_P^\lambda, \sqrt{1 - (1 - v_P^2)^\lambda}\right)$

Definition 2.4[13]

For any PFN the score function of P is defined as follows:

$$S(P) = \mu_P^2(x) - v_P^2(x)$$

where $S(P) \in [-1,1]$. For any two PFNs P_1, P_2 , if $S(P_1) < S(P_2)$, then $P_1 < P_2$. If $S(P_1) > S(P_2)$, then $P_1 > P_2$.If $S(P_1) = S(P_2)$, then $P_1 \sim P_2$.

Definition 2.5[13]

For any PFNs $P = (\mu_P, v_P)$, the accuracy function of A is defined as follows:

$$a(P) = \mu_P^2(x) + v_P^2(x)$$

where $a(P) \in [0,1]$.

3.Bipolar Pythagorean Fuzzy Sets

Definition 3.1 [11]

Let X be a non-empty set. A bipolar Pythagorean fuzzy set (BPFS) $\beta_j = \left\{\left(x, \left(u_{\beta_j}^P, v_{\beta_j}^P\right), \left(u_{\beta_j}^N, v_{\beta_j}^N\right)\right) \middle| x \in X\right\}$ where $u_{\beta_j}^P: X \to [0,1]$, $v_{\beta_j}^P: X \to [0,1]$, $u_{\beta_j}^N: X \to [-1,0]$, $v_{\beta_j}^N: X \to [-1,0]$ are the mappings such that

$$0 \le \left(u_{\beta_j}^P(x)\right)^2 + \left(v_{\beta_j}^P(x)\right)^2 \le 1 \text{ and } -1 \le -\left(\left(u_{\beta_j}^N(x)\right)^2 + \left(v_{\beta_j}^N(x)\right)^2\right) \le 0$$

and $u_{\beta_j}^P(x)$ denote the positive membership degree, $v_{\beta_j}^P(x)$ denote the positive non-membership degree, $u_{\beta_j}^N(x)$ denote the negative membership degree and $v_{\beta_j}^N(x)$ denote the negative non-membership degree. The degree of indeterminacy

$$\pi_{\beta_j}^P(x) = \sqrt{1 - \left(u_{\beta_j}^P(x)\right)^2 - \left(v_{\beta_j}^P(x)\right)^2} \text{ and } \pi_{\beta_j}^N(x) = -\sqrt{1 - \left(u_{\beta_j}^N(x)\right)^2 - \left(v_{\beta_j}^N(x)\right)^2}.$$

Where $\left(\pi_{\beta_j}^P(x)\right)^2 = 1 - \left(u_{\beta_j}^P(x)\right)^2 - \left(v_{\beta_j}^P(x)\right)^2$, $\left(\pi_{\beta_j}^N(x)\right)^2 = 1 - \left(u_{\beta_j}^N(x)\right)^2 - \left(v_{\beta_j}^N(x)\right)^2$, $0 \le \left(\pi_{\beta_j}^P(x)\right)^2 \le 1$,

$-1 \le -\left(\pi_{\beta_j}^N(x)\right)^2 \le 0.$

Definition 3.2 [11]

For any BPFN the score function of β is defined as follows:

$$S(\beta) = \frac{1}{2}\left(\left(u_\beta^P(x)\right)^2 - \left(v_\beta^P(x)\right)^2 + \left(u_\beta^N(x)\right)^2 - \left(v_\beta^N(x)\right)^2\right)$$

where $S(\beta) \in [-1,1]$. For any two BPFNs β_1, β_2, if $S(\beta_1) < S(\beta_2)$, then $\beta_1 < \beta_2$.If $S(\beta_1) > S(\beta_2)$, then $\beta_1 > \beta_2$.If $S(\beta_1) = S(\beta_2)$, then $\beta_1 \sim \beta_2$.

Definition 3.3 [11]

For any BPFNs $\beta = P\left(u_\beta^P, v_\beta^P, u_\beta^N, v_\beta^N\right)$, the accuracy function of A is defined as follows:

$$a(\beta) = \frac{1}{2}\left(\left(u_\beta^P(x)\right)^2 + \left(v_\beta^P(x)\right)^2 + \left(u_\beta^N(x)\right)^2 + \left(v_\beta^N(x)\right)^2\right)$$

where $a(\beta) \in [0,1]$.

Definition 3.4 [11]

Let $\beta_1 = \left\{\left(x, \left(u_{\beta_1}^P, v_{\beta_1}^P\right), \left(u_{\beta_1}^N, v_{\beta_1}^N\right)\right) : x \in X\right\}$ and $\beta_2 = \left\{\left(x, \left(u_{\beta_2}^P, v_{\beta_2}^P\right), \left(u_{\beta_2}^N, v_{\beta_2}^N\right)\right) : x \in X\right\}$be two BPFSs, then their operations are defined as follows:

1) $\beta_1 \cup \beta_2 = \left\{\left(x, max\left(u_{\beta_1}^P, u_{\beta_2}^P\right), min\left(v_{\beta_1}^P, v_{\beta_2}^P\right), min\left(u_{\beta_1}^N, u_{\beta_2}^N\right), max\left(v_{\beta_1}^N, v_{\beta_2}^N\right)\right) : x \in X\right\}$
2) $\beta_1 \cup \beta_2 = \left\{\left(x, min\left(u_{\beta_1}^P, u_{\beta_2}^P\right), max\left(v_{\beta_1}^P, v_{\beta_2}^P\right), max\left(u_{\beta_1}^N, u_{\beta_2}^N\right), min\left(v_{\beta_1}^N, v_{\beta_2}^N\right)\right) : x \in X\right\}$
3) $\beta^C = \left\{\left(x, \left(v_\beta^P, u_\beta^P\right), \left(v_\beta^N, u_\beta^N\right)\right) : x \in X\right\}$

Definition 3.5 [11]

Let $\beta_1 = P\left(u_{\beta_1}^P, v_{\beta_1}^P, u_{\beta_1}^N, v_{\beta_1}^N\right)$, and $\beta_2 = P\left(u_{\beta_2}^P, v_{\beta_2}^P, u_{\beta_2}^N, v_{\beta_2}^N\right)$, be two BPFNs and $\lambda > 0$, then their operations are defined as follows:

$$(1)\beta_1 \oplus \beta_2 = P\left(\left(\sqrt{\left(u_{\beta_1}^P\right)^2 + \left(u_{\beta_2}^P\right)^2 - \left(u_{\beta_1}^P\right)^2\left(u_{\beta_2}^P\right)^2}, v_{\beta_1}^P v_{\beta_2}^P\right), \left(-u_{\beta_1}^N u_{\beta_2}^N, -\sqrt{\left(v_{\beta_1}^N\right)^2 + \left(v_{\beta_2}^N\right)^2 - \left(v_{\beta_1}^N\right)^2\left(v_{\beta_2}^N\right)^2}\right)\right)$$

$$(2)\beta_1 \otimes \beta_2 = P\left(\left(u_{\beta_1}^P u_{\beta_2}^P, \sqrt{\left(v_{\beta_1}^P\right)^2 + \left(v_{\beta_2}^P\right)^2 - \left(v_{\beta_1}^P\right)^2\left(v_{\beta_2}^P\right)^2}\right), \left(-\sqrt{\left(u_{\beta_1}^N\right)^2 + \left(u_{\beta_2}^N\right)^2 - \left(u_{\beta_1}^N\right)^2\left(u_{\beta_2}^N\right)^2}, -v_{\beta_1}^N v_{\beta_2}^N\right)\right)$$

$$(3)\ \lambda\beta_1 = P\left(\left(\sqrt{1 - \left(1 - \left(u_{\beta_1}^P\right)^2\right)^\lambda}, \left(v_{\beta_1}^P\right)^\lambda\right), \left(-\left(-u_{\beta_1}^N\right)^\lambda, -\sqrt{1 - \left(1 - \left(v_{\beta_1}^N\right)^2\right)^\lambda}\right)\right)$$

(4) $A^{\lambda} = P\left(\left(\left(u_{\beta_1}^{P}\right)^{\lambda}, \sqrt{1-\left(1-\left(v_{\beta_1}^{P}\right)^2\right)^{\lambda}}\right), \left(-\sqrt{1-\left(1-\left(u_{\beta_1}^{N}\right)^2\right)^{\lambda}}, -\left(-v_{\beta_1}^{N}\right)^{\lambda}\right)\right)$

Theorem 3.1

Let $\beta_1 = P\left(u_{\beta_1}^{P}, v_{\beta_1}^{P}, u_{\beta_1}^{N}, v_{\beta_1}^{N}\right)$ and $\beta_2 = P\left(u_{\beta_2}^{P}, v_{\beta_2}^{P}, u_{\beta_2}^{N}, v_{\beta_2}^{N}\right)$ be two BPFNs and $\lambda > 0, \lambda_1 > 0, \lambda_2 > 0$, then,

1) $\beta_1 \oplus \beta_2 = \beta_2 \oplus \beta_1;$
2) $\beta_1 \otimes \beta_2 = \beta_2 \otimes \beta_1;$
3) $\lambda(\beta_1 \oplus \beta_2) = \lambda\,\beta_1 \oplus \lambda\,\beta_2;$
4) $\lambda_1\,\beta_1 \oplus \lambda_2\,\beta_1 = (\lambda_1+\lambda_2)\,\beta_1;$
5) $(\beta_1 \otimes \beta_2)^{\lambda} = {\beta_1}^{\lambda} \otimes {\beta_2}^{\lambda};$
6) ${\beta_1}^{\lambda_1} \otimes {\beta_1}^{\lambda_2} = {\beta_1}^{(\lambda_1+\lambda_2)};$

Proof:

For the three PFNs A,B and C and λ, λ_1 and $\lambda_2 > 0$, according to Definition 2.1, we can prove

$$1)\ \beta_1 \oplus \beta_2 = P\left(\left(\sqrt{\left(u_{\beta_1}^{P}\right)^2+\left(u_{\beta_2}^{P}\right)^2-\left(u_{\beta_1}^{P}\right)^2\left(u_{\beta_2}^{P}\right)^2}, v_{\beta_1}^{P}v_{\beta_2}^{P}\right), \left(-u_{\beta_1}^{N}u_{\beta_2}^{N}, -\sqrt{\left(v_{\beta_1}^{N}\right)^2+\left(v_{\beta_2}^{N}\right)^2-\left(v_{\beta_1}^{N}\right)^2\left(v_{\beta_2}^{N}\right)^2}\right)\right)$$

$$= P\left(\left(\sqrt{\left(u_{\beta_2}^{P}\right)^2+\left(u_{\beta_1}^{P}\right)^2-\left(u_{\beta_2}^{P}\right)^2\left(u_{\beta_1}^{P}\right)^2}, v_{\beta_2}^{P}v_{\beta_1}^{P}\right), \left(-u_{\beta_2}^{N}u_{\beta_1}^{N}, -\sqrt{\left(v_{\beta_2}^{N}\right)^2+\left(v_{\beta_1}^{N}\right)^2-\left(v_{\beta_2}^{N}\right)^2\left(v_{\beta_1}^{N}\right)^2}\right)\right)$$

$$= \beta_2 \oplus \beta_1.$$

$$2)\ \beta_1 \otimes \beta_2 = P\left(\left(u_{\beta_1}^{P}u_{\beta_2}^{P}, \sqrt{\left(v_{\beta_1}^{P}\right)^2+\left(v_{\beta_2}^{P}\right)^2-\left(v_{\beta_1}^{P}\right)^2\left(v_{\beta_2}^{P}\right)^2}\right), \left(-\sqrt{\left(u_{\beta_1}^{N}\right)^2+\left(u_{\beta_2}^{N}\right)^2-\left(u_{\beta_1}^{N}\right)^2\left(u_{\beta_2}^{N}\right)^2}, -v_{\beta_1}^{N}v_{\beta_2}^{N}\right)\right)$$

$$= P\left(\left(u_{\beta_2}^{P}u_{\beta_1}^{P}, \sqrt{\left(v_{\beta_2}^{P}\right)^2+\left(v_{\beta_1}^{P}\right)^2-\left(v_{\beta_2}^{P}\right)^2\left(v_{\beta_1}^{P}\right)^2}\right), \left(-\sqrt{\left(u_{\beta_2}^{N}\right)^2+\left(u_{\beta_1}^{N}\right)^2-\left(u_{\beta_2}^{N}\right)^2\left(u_{\beta_1}^{N}\right)^2}, -v_{\beta_2}^{N}v_{\beta_1}^{N}\right)\right)$$

$$= \beta_2 \otimes \beta_1.$$

$$3)\ \lambda(\beta_1 \oplus \beta_2) = \lambda\left(P\left(\left(\sqrt{\left(u_{\beta_1}^{P}\right)^2+\left(u_{\beta_2}^{P}\right)^2-\left(u_{\beta_1}^{P}\right)^2\left(u_{\beta_2}^{P}\right)^2}, v_{\beta_1}^{P}v_{\beta_2}^{P}\right),\right.\right.$$

$$\left.\left.\left(-u_{\beta_1}^{N}u_{\beta_2}^{N}, -\sqrt{\left(v_{\beta_1}^{N}\right)^2+\left(v_{\beta_2}^{N}\right)^2-\left(v_{\beta_1}^{N}\right)^2\left(v_{\beta_2}^{N}\right)^2}\right)\right)\right)$$

$$= P\left(\sqrt{1-\left(1-\left(\left(u_{\beta_1}^{P}\right)^2+\left(u_{\beta_2}^{P}\right)^2-\left(u_{\beta_1}^{P}\right)^2\left(u_{\beta_2}^{P}\right)^2\right)\right)^{\lambda}}, \left(v_{\beta_1}^{P}v_{\beta_2}^{P}\right)^{\lambda}, -\left(-\left(-u_{\beta_1}^{N}u_{\beta_2}^{N}\right)\right)^{\lambda},\right.$$

$$\left.-\sqrt{1-\left(1-\left(\left(v_{\beta_1}^{N}\right)^2+\left(v_{\beta_2}^{N}\right)^2-\left(v_{\beta_1}^{N}\right)^2\left(v_{\beta_2}^{N}\right)^2\right)^{\lambda}\right)}\right)$$

$$= P\left(\sqrt{1-\left(1-\left(\left(u_{\beta_1}^{P}\right)^2+\left(u_{\beta_2}^{P}\right)^2-\left(u_{\beta_1}^{P}\right)^2\left(u_{\beta_2}^{P}\right)^2\right)\right)^{\lambda}}, \left(v_{\beta_1}^{P}v_{\beta_2}^{P}\right)^{\lambda}, -\left(u_{\beta_1}^{N}u_{\beta_2}^{N}\right)^{\lambda},\right.$$

$$\left.-\sqrt{1-\left(1-\left(\left(v_{\beta_1}^{N}\right)^2+\left(v_{\beta_2}^{N}\right)^2-\left(v_{\beta_1}^{N}\right)^2\left(v_{\beta_2}^{N}\right)^2\right)^{\lambda}\right)}\right)$$

$$\lambda\beta_1 \oplus \lambda\beta_2 = P\left(\left(\sqrt{1-\left(1-\left(u_{\beta_1}^{P}\right)^2\right)^{\lambda}}, {v_{\beta_1}^{P}}^{\lambda}, -\left(-u_{\beta_1}^{N}\right)^{\lambda}, -\sqrt{1-\left(1-\left(v_{\beta_1}^{N}\right)^2\right)^{\lambda}}\right) \oplus\right.$$

$$\left(\sqrt{1-\left(1-\left(u_{\beta_2}^P\right)^2\right)^\lambda}, {v_{\beta_2}^P}^\lambda, -\left(-u_{\beta_2}^N\right)^\lambda, -\sqrt{1-\left(1-\left(v_{\beta_2}^N\right)^2\right)^\lambda}\right)\Bigg)$$

$$= P\left(\left(\left(\sqrt{1-\left(1-\left(u_{\beta_1}^P\right)^2\right)^\lambda + 1-\left(1-\left(u_{\beta_2}^P\right)^2\right)^\lambda - \left(1-\left(1-\left(u_{\beta_1}^P\right)^2\right)^\lambda\right)\left(1-\left(1-\left(u_{\beta_2}^P\right)^2\right)^\lambda\right)}, \left(v_{\beta_1}^P v_{\beta_2}^P\right)^\lambda\right)\right.\right.$$

$$\left.\left.\left(-\left(u_{\beta_1}^N u_{\beta_2}^N\right)^\lambda, -\sqrt{1-\left(1-\left(v_{\beta_1}^N\right)^2\right)^\lambda + 1-\left(1-\left(v_{\beta_2}^N\right)^2\right)^\lambda - \left(1-\left(1-\left(v_{\beta_1}^N\right)^2\right)^\lambda\right)\left(1-\left(1-\left(v_{\beta_2}^N\right)^2\right)^\lambda\right)}\right)\right)\right)$$

$$= P\left(\sqrt{1-\left(1-\left(\left(u_{\beta_1}^P\right)^2+\left(u_{\beta_2}^P\right)^2-\left(u_{\beta_1}^P\right)^2\left(u_{\beta_2}^P\right)^2\right)\right)^\lambda}, \left(v_{\beta_1}^P v_{\beta_2}^P\right)^\lambda, -\left(u_{\beta_1}^N u_{\beta_2}^N\right)^\lambda,\right.$$

$$\left. -\sqrt{1-\left(1-\left(\left(v_{\beta_1}^N\right)^2+\left(v_{\beta_2}^N\right)^2-\left(v_{\beta_1}^N\right)^2\left(v_{\beta_2}^N\right)^2\right)^\lambda\right)}\right)$$

$$= \lambda(\beta_1 \oplus \beta_2).$$

(4) $$\lambda_1\beta \oplus \lambda_2\beta = P\left(\left(\sqrt{1-\left(1-\left(u_\beta^P\right)^2\right)^{\lambda_1}}, {v_\beta^P}^{\lambda_1}, -\left(-u_\beta^N\right)^{\lambda_1}, -\sqrt{1-\left(1-\left(v_\beta^N\right)^2\right)^{\lambda_1}}\right) \oplus\right.$$

$$\left.\left(\sqrt{1-\left(1-\left(u_\beta^P\right)^2\right)^{\lambda_2}}, {v_\beta^P}^{\lambda_2}, -\left(-u_\beta^N\right)^{\lambda_2}, -\sqrt{1-\left(1-\left(v_\beta^N\right)^2\right)^{\lambda_2}}\right)\right)$$

$$= P\left(\left(\left(\sqrt{1-\left(1-\left(u_\beta^P\right)^2\right)^{\lambda_1} + 1-\left(1-\left(u_\beta^P\right)^2\right)^{\lambda_2} - \left(1-\left(1-\left(u_\beta^P\right)^2\right)^{\lambda_1}\right)\left(1-\left(1-\left(u_\beta^P\right)^2\right)^{\lambda_2}\right)}, {v_\beta^P}^{\lambda_1}{v_\beta^P}^{\lambda_2}\right)\right.\right.$$

$$\left.\left.\left(-\left(-u_\beta^N\right)^{\lambda_1}\left(-u_\beta^N\right)^{\lambda_2}, -\sqrt{1-\left(1-\left(v_{\beta_1}^N\right)^2\right)^\lambda + 1-\left(1-\left(v_{\beta_1}^N\right)^2\right)^\lambda - \left(1-\left(1-\left(v_{\beta_1}^N\right)^2\right)^\lambda\right)\left(1-\left(1-\left(v_{\beta_2}^N\right)^2\right)^\lambda\right)}\right)\right)\right)$$

$$= P\left(\left(\sqrt{1-\left(1-\left(u_\beta^P\right)^2\right)^{\lambda_1+\lambda_2}}, \left(v_\beta^P\right)^{\lambda_1+\lambda_2}\right), \left(-\left(-u_\beta^N\right)^{\lambda_1+\lambda_2}, -\sqrt{1-\left(1-\left(v_\beta^N\right)^2\right)^{\lambda_1+\lambda_2}}\right)\right) = (\lambda_1+\lambda_2)\beta.$$

(5) $$(\beta_1 \otimes \beta_2)^\lambda = P\left(\left(\sqrt{\left(u_{\beta_1}^P\right)^2+\left(u_{\beta_2}^P\right)^2-\left(u_{\beta_1}^P\right)^2\left(u_{\beta_2}^P\right)^2}, v_{\beta_1}^P v_{\beta_2}^P\right), \left(-u_{\beta_1}^N u_{\beta_2}^N, -\sqrt{\left(v_{\beta_1}^N\right)^2+\left(v_{\beta_2}^N\right)^2-\left(v_{\beta_1}^N\right)^2\left(v_{\beta_2}^N\right)^2}\right)\right)^\lambda$$

$$= P\left(\left(u_{\beta_1}^P u_{\beta_2}^P\right)^\lambda, \sqrt{1-\left(1-\left(\left(v_{\beta_1}^P\right)^2+\left(v_{\beta_2}^P\right)^2-\left(v_{\beta_1}^P\right)^2\left(v_{\beta_2}^P\right)^2\right)\right)^\lambda},\right.$$

$$\left. -\sqrt{1-\left(1-\left(\left(u_{\beta_1}^N\right)^2+\left(u_{\beta_2}^N\right)^2-\left(u_{\beta_1}^N\right)^2\left(u_{\beta_2}^N\right)^2\right)^\lambda\right)}, -\left(-\left(-v_{\beta_1}^N v_{\beta_2}^N\right)\right)^\lambda\right)$$

$$= P\left(\left(u_{\beta_1}^P u_{\beta_2}^P\right)^\lambda, \sqrt{1-\left(1-\left(\left(v_{\beta_1}^P\right)^2+\left(v_{\beta_2}^P\right)^2-\left(v_{\beta_1}^P\right)^2\left(v_{\beta_2}^P\right)^2\right)\right)^\lambda},\right.$$

$$\left. -\sqrt{1-\left(1-\left(\left(u_{\beta_1}^N\right)^2+\left(u_{\beta_2}^N\right)^2-\left(u_{\beta_1}^N\right)^2\left(u_{\beta_2}^N\right)^2\right)^\lambda\right)}, -\left(v_{\beta_1}^N v_{\beta_2}^N\right)^\lambda\right)$$

$$\beta_1{}^{\lambda} \otimes \beta_2{}^{\lambda} = P\left(\left(u_{\beta_1}^{P\ \lambda}, \sqrt{1-\left(1-\left(v_{\beta_1}^{P}\right)^2\right)^{\lambda}}, -\sqrt{1-\left(1-\left(u_{\beta_1}^{N}\right)^2\right)^{\lambda}}, -\left(-v_{\beta_1}^{N}\right)^{\lambda}\right) \oplus\right.$$

$$\left.\left(u_{\beta_2}^{P\ \lambda}, \sqrt{1-\left(1-\left(v_{\beta_2}^{P}\right)^2\right)^{\lambda}}, -\sqrt{1-\left(1-\left(u_{\beta_2}^{N}\right)^2\right)^{\lambda}}, -\left(-v_{\beta_2}^{N}\right)^{\lambda}\right)\right)$$

$$= P\left(\left(\left(u_{\beta_1}^{P} u_{\beta_2}^{P}\right)^{\lambda}, \sqrt{1-\left(1-\left(v_{\beta_1}^{P}\right)^2\right)^{\lambda} + 1-\left(1-\left(v_{\beta_1}^{P}\right)^2\right)^{\lambda} - \left(1-\left(1-\left(v_{\beta_1}^{P}\right)^2\right)^{\lambda}\right)\left(1-\left(1-\left(v_{\beta_2}^{P}\right)^2\right)^{\lambda}\right)}\right)\right.$$

$$\left.\left(-\sqrt{1-\left(1-\left(u_{\beta_1}^{N}\right)^2\right)^{\lambda} + 1-\left(1-\left(u_{\beta_1}^{N}\right)^2\right)^{\lambda} - \left(1-\left(1-\left(u_{\beta_1}^{N}\right)^2\right)^{\lambda}\right)\left(1-\left(1-\left(u_{\beta_2}^{N}\right)^2\right)^{\lambda}\right)}, -\left(v_{\beta_1}^{N} v_{\beta_2}^{N}\right)^{\lambda}\right)\right)$$

$$= P\left(\left(u_{\beta_1}^{P} u_{\beta_2}^{P}\right)^{\lambda}, \sqrt{1-\left(1-\left(\left(v_{\beta_1}^{P}\right)^2 + \left(v_{\beta_2}^{P}\right)^2 - \left(v_{\beta_1}^{P}\right)^2\left(v_{\beta_2}^{P}\right)^2\right)\right)^{\lambda}},\right.$$

$$\left.-\sqrt{1-\left(1-\left(\left(u_{\beta_1}^{N}\right)^2 + \left(u_{\beta_2}^{N}\right)^2 - \left(u_{\beta_1}^{N}\right)^2\left(u_{\beta_2}^{N}\right)^2\right)^{\lambda}\right)}, -\left(v_{\beta_1}^{N} v_{\beta_2}^{N}\right)^{\lambda}\right)$$

$$= (\beta_1 \oplus \beta_2)^{\lambda}.$$

(6) $$\beta^{\lambda_1} \otimes \beta^{\lambda_2} = P\left(\left(u_{\beta}^{P\ \lambda_1}, \sqrt{1-\left(1-\left(v_{\beta}^{P}\right)^2\right)^{\lambda_1}}, -\sqrt{1-\left(1-\left(u_{\beta}^{N}\right)^2\right)^{\lambda_1}}, -\left(-v_{\beta}^{N}\right)^{\lambda_1}\right) \oplus\right.$$

$$\left.\left(u_{\beta}^{P\ \lambda_2}, \sqrt{1-\left(1-\left(v_{\beta}^{P}\right)^2\right)^{\lambda_2}}, -\sqrt{1-\left(1-\left(u_{\beta}^{N}\right)^2\right)^{\lambda_2}}, -\left(-v_{\beta}^{N}\right)^{\lambda_2}\right)\right)$$

$$= P\left(\left(u_{\beta}^{P\ \lambda_1} u_{\beta}^{P\ \lambda_2}, \sqrt{1-\left(1-\left(v_{\beta}^{P}\right)^2\right)^{\lambda_1} + 1-\left(1-\left(v_{\beta}^{P}\right)^2\right)^{\lambda_2} - \left(1-\left(1-\left(v_{\beta}^{P}\right)^2\right)^{\lambda_1}\right)\left(1-\left(1-\left(v_{\beta}^{P}\right)^2\right)^{\lambda_2}\right)}\right)\right.$$

$$\left.\left(-\sqrt{1-\left(1-\left(u_{\beta_1}^{N}\right)^2\right)^{\lambda} + 1-\left(1-\left(u_{\beta_1}^{N}\right)^2\right)^{\lambda} - \left(1-\left(1-\left(u_{\beta_1}^{N}\right)^2\right)^{\lambda}\right)\left(1-\left(1-\left(u_{\beta_2}^{N}\right)^2\right)^{\lambda}\right)}, -\left(-v_{\beta}^{N}\right)^{\lambda_1}\left(-v_{\beta}^{N}\right)^{\lambda_2}\right)\right)$$

$$= P\left(\left(\left(u_{\beta}^{P}\right)^{\lambda_1+\lambda_2}, \sqrt{1-\left(1-\left(v_{\beta}^{P}\right)^2\right)^{\lambda_1+\lambda_2}}\right), \left(-\sqrt{1-\left(1-\left(u_{\beta}^{N}\right)^2\right)^{\lambda_1+\lambda_2}}, -\left(-v_{\beta}^{N}\right)^{\lambda_1+\lambda_2}\right)\right) = \beta^{(\lambda_1+\lambda_2)}.$$

4. TOPSIS approach to MCDM problem with bipolar Pythagorean fuzzy information

This section first introduces the MCDM problem under bipolar Pythagorean fuzzy environment. Then, an effective decision-making approach is proposed to deal with such MCDM problems.

4.1 Description of the MCDM problem with BPFNs

A MCDM problem can be expressed as a decision matrix whose elements indicate the evaluation values of all alternatives with respect to each criterion. For a given MCDM problem under bipolar Pythagorean fuzzy environment, let $X = \{x_1, x_2, \dots, x_m\}$ $(m \geq 2)$ be a discrete of m feasible alternatives, $C = \{c_1, c_2, \dots, c_m\}$ be a finite set of criteria, and $\omega = (\omega_1, \omega_2, \dots, \omega_m)^T$ be the weight vector of all criteria,which satisfy $0 \leq w_j \leq 1$ and $\sum_{j=1}^{n} \omega_j = 1$.We denote the evaluation values of the alternative $x_i (i = 1,2, \dots, m)$ with respect to the criteria $C_j (j = 1,2, \dots, n)$ by $C_j(x_i) = P\left(u_{ij}^{P}, v_{ij}^{P}, u_{ij}^{N}, v_{ij}^{N}\right)$ and $R =$

$\left(C_j(x_i)\right)_{m\times n}$ is a bipolar Pythagorean fuzzy decision matrix. Therefore, the MCDM problem with BPFNs can be represented as the following matrix form:

$$R=\left(C_j(x_i)\right)_{m\times n}=\begin{matrix} \\ x_1 \\ \vdots \\ x_m\end{matrix}\begin{matrix} C_1 & \dots & C_n \\ \begin{pmatrix} P(u_{11}^P,v_{11}^P,u_{11}^N,v_{11}^N) & \cdots & P(u_{1n}^P,v_{1n}^P,u_{1n}^N,v_{1n}^N) \\ \vdots & \ddots & \vdots \\ P(u_{m1}^P,v_{m1}^P,u_{m1}^N,v_{m1}^N) & \cdots & P(u_{mn}^P,v_{mn}^P,u_{mn}^N,v_{mn}^N)\end{pmatrix}\end{matrix} \quad (4.1)$$

where each of elements $C_j(x_i)=P\left(u_{ij}^P,v_{ij}^P,u_{ij}^N,v_{ij}^N\right)$ is a BPFN.

4.2 The Proposed Decision Approach

To effectively solve the aforementioned MCDM problem with BPFNs, in the following we propose a bipolar Pythagorean fuzzy TOPSIS method. The proposed method is based on the principle that the optimal alternative should have the shortest distance from the PIS and the farthest distance from the NIS.

Therefore, this approach starts with the determination of the bipolar Pythagorean fuzzy PIS and the bipolar Pythagorean fuzzy NIS. Considering that the decision information takes the form of BPFNs, we utilize the score function based comparison approach introduced in Definition 2.4 to identify the bipolar Pythagorean fuzzy PIS and the Pythagorean fuzzy NIS. We denote the bipolar Pythagorean fuzzy PIS by x^+,which can be determined by the following formula:

$$x^+ = \{C_j, max\left(u_{ij}^P\right)|, min\left(v_{ij}^P\right), min\left(u_{ij}^N\right), max\left(v_{ij}^N\right)|j=1,2,\dots,n\}$$

$$=\{\left(C_1,P((u_1^P)^+,(v_1^P)^+,(u_1^N)^+,(v_1^N)^+)\right)\left(C_2,P((u_2^P)^+,(v_2^P)^+,(u_2^N)^+,(v_2^N)^+)\right),\dots,$$

$$\left(C_n,P((u_n^P)^+,(v_n^P)^+,(u_n^N)^+,(v_n^N)^+)\right)\} \quad (4.2)$$

In the real –life MCDM process, there usually exist no bipolar Pythagorean fuzzy PIS. In other words, the bipolar Pythagorean fuzzy PIS x^+ is usual not be the feasible alternative, namely, $x^+\notin X$. Otherwise, the bipolar Pythagorean fuzzy PIS x^+ is the optimal alternative of the MCDM problem. Then, we proceed to calculate the distance between each alternative and the bipolar Pythagorean fuzzy PIS.To this end, we need to define the concept of distance measure for BPFNs.

Definition 4.1

Let $\beta_j=\left(u_{\beta_j}^P,v_{\beta_j}^P,u_{\beta_j}^N,v_{\beta_j}^N\right)$ $(j=1,2)$ be two BPFNs, then we define the distance between β_1 and β_2 as follows:

$$d(\beta_1,\beta_2)=\frac{1}{4}\Big(\left|\left(u_{\beta_1}^P\right)^2-\left(u_{\beta_2}^P\right)^2\right|+\left|\left(v_{\beta_1}^P\right)^2-\left(v_{\beta_2}^P\right)^2\right|+\left|\left(u_{\beta_1}^N\right)^2-\left(u_{\beta_2}^N\right)^2\right|+\left|\left(v_{\beta_1}^N\right)^2-\left(v_{\beta_2}^N\right)^2\right|+ \left|\left(\pi_{\beta_1}^P\right)^2-\left(\pi_{\beta_2}^P\right)^2\right|+\left|\left(\pi_{\beta_1}^N\right)^2-\left(\pi_{\beta_2}^N\right)^2\right|\Big) \quad (4.3)$$

Theorem 4.1

Let $\beta_j=\left(u_{\beta_j}^P,v_{\beta_j}^P,u_{\beta_j}^N,v_{\beta_j}^N\right)$ $(j=1,2)$ be two BPFNs, then $0\le d(\beta_1,\beta_2)\le 1$.

Proof:

Because $0\le u_{\beta_1}^P,v_{\beta_1}^P,u_{\beta_2}^P,v_{\beta_2}^P\le 1,-1\le u_{\beta_1}^N,v_{\beta_1}^N,u_{\beta_2}^N,v_{\beta_2}^N\le 0$,

$\left(u_{\beta_1}^P\right)^2+\left(v_{\beta_1}^P\right)^2\le 1$, $\left(u_{\beta_2}^P\right)^2+\left(v_{\beta_2}^P\right)^2\le 1$,

$-\left(\left(u_{\beta_1}^N\right)^2+\left(v_{\beta_1}^N\right)^2\right)\ge -1$ and $-\left(\left(u_{\beta_2}^N\right)^2+\left(v_{\beta_2}^N\right)^2\right)\ge -1$, then

$$\begin{aligned} d(\beta_1,\beta_2)&=\tfrac{1}{4}\Big(\left|\left(u_{\beta_1}^P\right)^2-\left(u_{\beta_2}^P\right)^2\right|+\left|\left(v_{\beta_1}^P\right)^2-\left(v_{\beta_2}^P\right)^2\right|+\left|\left(u_{\beta_1}^N\right)^2-\left(u_{\beta_2}^N\right)^2\right|+\left|\left(v_{\beta_1}^N\right)^2-\left(v_{\beta_2}^N\right)^2\right|+ \\ &\left|1-\left(u_{\beta_1}^P\right)^2-\left(v_{\beta_1}^P\right)^2-\left(1-\left(u_{\beta_2}^P\right)^2-\left(v_{\beta_2}^P\right)^2\right)\right|+\left|1-\left(u_{\beta_1}^P\right)^2-\left(v_{\beta_1}^P\right)^2-\left(1-\left(u_{\beta_2}^P\right)^2-\left(v_{\beta_2}^P\right)^2\right)\right|\Big) \\ &=\tfrac{1}{4}\Big(\left|\left(u_{\beta_1}^P\right)^2-\left(u_{\beta_2}^P\right)^2\right|+\left|\left(v_{\beta_1}^P\right)^2-\left(v_{\beta_2}^P\right)^2\right|+\left|\left(u_{\beta_1}^N\right)^2-\left(u_{\beta_2}^N\right)^2\right|+\left|\left(v_{\beta_1}^N\right)^2-\left(v_{\beta_2}^N\right)^2\right|+ \\ &\left|\left(u_{\beta_2}^P\right)^2-\left(u_{\beta_1}^P\right)^2+\left(v_{\beta_2}^P\right)^2-\left(v_{\beta_1}^P\right)^2\right|+\left|\left(u_{\beta_2}^N\right)^2-\left(u_{\beta_1}^N\right)^2+\left(v_{\beta_2}^N\right)^2-\left(v_{\beta_1}^N\right)^2\right|\Big) \\ &\le\tfrac{1}{4}\left(\left(u_{\beta_1}^P\right)^2+\left(v_{\beta_1}^P\right)^2+\left(u_{\beta_2}^P\right)^2+\left(v_{\beta_2}^P\right)^2+\left(u_{\beta_1}^N\right)^2+\left(v_{\beta_1}^N\right)^2+\left(u_{\beta_2}^N\right)^2+\left(v_{\beta_2}^N\right)^2\right) \\ &\le\tfrac{1}{4}(4)=1\end{aligned}$$

Additionally, according to Definition 4.1,it can be easily seen that $d(\beta_1,\beta_2) \geq 0$. Thus, $0 \leq d(\beta_1,\beta_2) \leq 1$, which completes the proof of Theorem 4.1.

Theorem 4.2

Let $\beta_j = \left(u_{\beta_j}^P, v_{\beta_j}^P, u_{\beta_j}^N, v_{\beta_j}^N\right)$ $(j = 1,2)$ be two BPFNs, then $d(\beta_1,\beta_2) = 0$, if and only if $\beta_1 = \beta_2$.

Theorem 4.3

Let $\beta_j = \left(u_{\beta_j}^P, v_{\beta_j}^P, u_{\beta_j}^N, v_{\beta_j}^N\right)$ $(j = 1,2)$ be two BPFNs, then $d(\beta_1,\beta_2) = d(\beta_2,\beta_1)$.

Proof:

$$d(\beta_1,\beta_2) = \frac{1}{4}\left(\left|\left(u_{\beta_1}^P\right)^2 - \left(u_{\beta_2}^P\right)^2\right| + \left|\left(v_{\beta_1}^P\right)^2 - \left(v_{\beta_2}^P\right)^2\right| + \left|\left(u_{\beta_1}^N\right)^2 - \left(u_{\beta_2}^N\right)^2\right| + \left|\left(v_{\beta_1}^N\right)^2 - \left(v_{\beta_2}^N\right)^2\right| + \left|1 - \left(u_{\beta_1}^P\right)^2 - \left(v_{\beta_1}^P\right)^2 - \left(1 - \left(u_{\beta_2}^P\right)^2 - \left(v_{\beta_2}^P\right)^2\right)\right| + \left|1 - \left(u_{\beta_1}^P\right)^2 - \left(v_{\beta_1}^P\right)^2 - \left(1 - \left(u_{\beta_2}^P\right)^2 - \left(v_{\beta_2}^P\right)^2\right)\right|\right)$$

$$= \frac{1}{4}\left(\left|\left(u_{\beta_2}^P\right)^2 - \left(u_{\beta_1}^P\right)^2\right| + \left|\left(v_{\beta_2}^P\right)^2 - \left(v_{\beta_1}^P\right)^2\right| + \left|\left(u_{\beta_2}^N\right)^2 - \left(u_{\beta_1}^N\right)^2\right| + \left|\left(v_{\beta_2}^N\right)^2 - \left(v_{\beta_1}^N\right)^2\right| + \left|1 - \left(u_{\beta_2}^P\right)^2 - \left(v_{\beta_2}^P\right)^2 - \left(1 - \left(u_{\beta_1}^P\right)^2 - \left(v_{\beta_1}^P\right)^2\right)\right| + \left|1 - \left(u_{\beta_2}^P\right)^2 - \left(v_{\beta_2}^P\right)^2 - \left(1 - \left(u_{\beta_1}^P\right)^2 - \left(v_{\beta_1}^P\right)^2\right)\right|\right)$$

$$= d(\beta_2,\beta_1).$$

Thus, the distance between the alternative x_i and the bipolar Pythagorean fuzzy PIS x^+ can be calculated by using Equation (4.3) as follows:

$$D(x_i,x^+) = \sum_{j=1}^{n} \omega_j d\left(C_j(x_i), C_j(x^+)\right)$$

$$= \frac{1}{4}\sum_{j=1}^{n} \omega_j \left(\left|\left(u_{ij}^P\right)^2 - \left(\left(u_j^P\right)^+\right)^2\right| + \left|\left(v_{ij}^P\right)^2 - \left(\left(v_j^P\right)^+\right)^2\right| + \left|\left(u_{ij}^N\right)^2 - \left(\left(u_j^N\right)^+\right)^2\right| + \left|\left(v_{ij}^N\right)^2 - \left(\left(v_j^N\right)^+\right)^2\right| + \left|\left(\pi_{ij}^P\right)^2 - \left(\left(\pi_j^P\right)^+\right)^2\right| + \left|\left(\pi_{ij}^N\right)^2 - \left(\left(\pi_j^N\right)^+\right)^2\right|\right)$$

$$i = 1,2, \dots, n \qquad (4.4)$$

Usually, the smaller $D(x_i,x^+)$ the better the alternative x_i and let

$$D_{min}(x_i,x^+) = \min_{1\leq i\leq m} D(x_i,x^+) \qquad (4.5)$$

However, the alternative with the closest distance to bipolar Pythagorean fuzzy PIS may be not the farthest from bipolar Pythagorean fuzzy NIS. We denote the bipolar Pythagorean fuzzy NIS by x^-, which can be determined by the following formula:

$$x^- = \{C_j, min(u_{ij}^P)|, max(v_{ij}^P), max(u_{ij}^N), min(v_{ij}^N)|j = 1,2, \dots, n\}$$

$$= \left\{\left(C_1, P\left(\left(u_1^P\right)^-, \left(v_1^P\right)^-, \left(u_1^N\right)^-, \left(v_1^N\right)^-\right)\right)\left(C_2, P\left(\left(u_2^P\right)^-, \left(v_2^P\right)^-, \left(u_2^N\right)^-, \left(v_2^N\right)^-\right)\right), \dots, \left(C_n, P((u_n^P)^-, (v_n^P)^-, (u_n^N)^-, (v_n^N)^-)\right)\right\} \qquad (4.6)$$

It is easily seen from Equation (4.6) that the obtained value of bipolar Pythagorean fuzzy NIS under each criterion is minimal among all the alternatives. Usually, in the practical MCDM process, there may not exist the bipolar Pythagorean fuzzy NIS. In other words, the bipolar Pythagorean fuzzy NIS x^- is usually an unfeasible alternative, namely, $x^- \notin X$. Otherwise, the bipolar Pythagorean fuzzy NIS x^- is the worst alternative of the MCDM problem, which should be deleted in the decision process.

Using this Equation (4.3), the distance between the alternative x_i and the bipolar Pythagorean fuzzy NIS x^- can be obtained as follows:

$$D(x_i,x^-) = \sum_{j=1}^{n} \omega_j d\left(C_j(x_i), C_j(x^-)\right)$$

$$= \frac{1}{4}\sum_{j=1}^{n} \omega_j \left(\left|\left(u_{ij}^P\right)^2 - \left(\left(u_j^P\right)^-\right)^2\right| + \left|\left(v_{ij}^P\right)^2 - \left(\left(v_j^P\right)^-\right)^2\right| + \left|\left(u_{ij}^N\right)^2 - \left(\left(u_j^N\right)^-\right)^2\right| + \left|\left(v_{ij}^N\right)^2 - \left(\left(v_j^N\right)^-\right)^2\right| + \left|\left(\pi_{ij}^P\right)^2 - \left(\left(\pi_j^P\right)^-\right)^2\right| + \left|\left(\pi_{ij}^N\right)^2 - \left(\left(\pi_j^N\right)^-\right)^2\right|\right)$$

$$i = 1,2, \dots, m \qquad (4.7)$$

In general, the bigger $D(x_i,x^-)$ the better the alternative x_i , and let

$$D_{max}(x_i, x^-) = \max_{1 \le i \le m} D(x_i, x^-) \quad (4.8)$$

In the classical TOPSIS method, we usually need to calculate the relative closeness of the alternative x_i with respect to the bipolar Pythagorean fuzzy PIS x^+ as below;

$$RC(x_i) = \frac{D(x_i, x^-)}{D(x_i, x^+) + D(x_i, x^-)} \quad (4.9)$$

According to the closeness index $RC(x_i)$, the ranking orders of all alternatives and the optimal alternatives can be determined. However, Hadi-Vencheh and Mirjaberi showed that in some situations, the relative closeness cannot achieve the aim that the optimal solution should have the shortest distance from the PIS and the farthest distance from the NIS, simultaneously. Thus, they suggested that one may use the following formula instead of the relative closeness index(i.e., Equation(4.9);

$$\zeta(x_i) = \frac{D(x_i, x^-)}{D_{max}(x_i, x^-)} - \frac{D(x_i, x^+)}{D_{min}(x_i, x^+)} \quad (4.10)$$

which is called the revised closeness used to measure the extent to which the alternative x_i is close to the bipolar Pythagorean fuzzy PIS x^+ and is far away from the bipolar Pythagorean fuzzy NIS x^-, simultaneously.

It can be easily seen that $\zeta(x_i) \le 0$ $(i = 1,2, \dots, m)$ and the bigger $\zeta(x_i)$, the better the alternative x_i .If there exists an alternative x^* satisfying the conditions that $D(x^*, x^-) = D_{max}(x^*, x^-)$ and $D(x^*, x^+) = D_{min}(x^*, x^+)$, simultaneously, then $\zeta(x^*) = 0$ and, obviously, the alternative x^* is the best alternative that is closest to the bipolar Pythagorean fuzzy PIS x^+ and farthest away from the Pythagorean fuzzy NIS x^-,simultaneously.

4.3 The algorithm of the Proposed method

On the basis of above analysis, we present a practical algorithm involves the following steps:

Step 1: For a MCDM problem with BPFNs, we construct the decision matrix $R = \left(C_j(x_i)\right)_{m \times n}$ where the elements $C_j(x_i)$ $(i = 1,2, \dots, m, j = 1,2, \dots, n)$ are the assessments of the alternative $x_i \in X$ with respect to the criterion $C_j \in C$.

Step 2: Employ Equations (4.2) and (4.6) to identify the bipolar Pythagorean fuzzy PIS= $\{C_1(x^+), C_2(x^+), \dots, C_n(x^+)\}$ and the bipolar Pythagorean fuzzy NIS $= \{C_1(x^-), C_2(x^-), \dots, C_n(x^-)\}$, respectively.

Step 3: Use Equations (4.4) and (4.7) to calculate the distances between the alternative x_i and the bipolar Pythagorean fuzzy PIS x^+ as well as the bipolar Pythagorean fuzzy NIS x^- , respectively.

Step 4: Utilize Equation (4.10) to calculate the revised closeness $\zeta(x_i)$ of the alternative x_i (i=1,2,…,m).

Step 5: Determine the optimal ranking order of the alternatives and identify the optimal alternative.On the basis of the revised closeness $\zeta(x_i)$ obtained from Step 4, we put the alternatives into orders with respect to the decreasing values of $\zeta(x_i)$ $(i = 1,2, \dots, m)$ and the alternative with the maximal revised closeness $\zeta(x_i)$ is the best alternative, namely.

$$x^* \coloneqq \left\{x_i: \left(i: \zeta(x_i) = \max_{1 \le i \le m} \zeta(x_i)\right)\right\} \quad (4.11)$$

5. ILLUSTRATION EXAMPLE

In this section, we consider a decision-making problem that concerns the evaluation of the cutting machine to illustrate the proposed approach .

Our aim is to provide support for decision makers regarding cutting-machine purchase in Erfe Shoe-Making Ind. Trade Co. Ltd., using TOPSIS method. The firm believes that purchasing a cutting machine will improve efficiency. The firm is considering offers from four firms, named A, B, C and D. Machine selection varies from company to company according to the type of work done by the company, expectations from the machine, and the purpose of the machine will serve. In this regard, criteria affecting the purchase decision were determined after interviews with the vice general manager, industrial relations expert and the foreman who will use. The criteria were identified as follows: Cost, service support, rate of waste, energy consumption.

Table I. Bipolar Pythagorean Fuzzy Decision Matrix

	C_1	C_2	C_3	C_4
x_1	$(0.9,0.3)(-0.2,-0.8)$	$(0.7,0.6)(-0.3,-0.5)$	$(0.5,0.8)(-0.8,-0.1)$	$(0.1,0.9)(-0.5,-0.3)$
x_2	$(0.7,0.4)(-0.6,-0.4)$	$(0.8,0.3)(-0.1,-0.7)$	$(0.9,0.1)(-0.7,-0.4)$	$(0.5,0.3)(-0.2,-0.5)$
x_3	$(0.6,0.1)(-0.8,-0.2)$	$(0.5,0.6)(-0.4,-0.5)$	$(0.3,0.6)(-0.5,-0.4)$	$(0.7,0.5)(-0.2,-0.8)$
x_4	$(0.8,0.5)(-0.2,-0.7)$	$(0.6,0.2)(-0.6,-0.1)$	$(0.2,0.8)(-0.6,-0.3)$	$(0.4,0.8)(-0.7,-0.3)$

5.1 Description

The weight vector of the criteria is given by the committee as $W = (0.50,0.25,0.125,0.125)^T$.Assume that the assessment values of the alternatives with respect to each criteria provided by the committee are represented by BPFNs as shown in the bipolar Pythagorean fuzzy decision matrix given in Table II.

	$D(x_i,x^+)$	$D(x_i,x^-)$	$\zeta(x_i)$	Ranking
x_1	0.3431	0.2444	-0.5445	4
x_2	0.2688	0.3338	0.0000	1
x_3	0.3281	0.2934	-0.3419	2
x_4	0.3450	0.2556	-0.5180	3

5.2 Decision Process

In the following, we use the bipolar Pythagorean fuzzy TOPSIS approach to solve the decision problem mentioned in Section 5.1.First, we utilize Equations (4.2) and (4.6) to determine the bipolar Pythagorean fuzzy PIS x^+ and the bipolar Pythagorean fuzzy NIS x^- , respectively, and the results are obtained as follows:

$$x^+ = \{P(0.9,0.1,-0.8,-0.2), P(0.8,0.2,-0.6,-0.1), P(0.9,0.1,-0.8,-0.1), P(0.7,0.3,-0.7,-0.3)\}$$

$$x^- = \{P(0.6,0.5,-0.2,-0.8), P(0.5,0.6,-0.1,-0.7), P(0.2,0.8,-0.5,-0.4), P(0.1,0.9,-0.2,-0.8)\}$$

Then, we employ Equations (4.4) and (4.7) to calculate the distances between the alternative x_i and bipolar Pythagorean fuzzy PIS x^+ as well as the bipolar Pythagorean fuzzy PIS x^- ,respectively.The results are shown in Table III.Moreover, we utilize Equation (4.10) to calculate the revised closeness $\zeta(x_i)$ of the alternative x_i, and the results are also listed in Table III.According to $\zeta(x_i)$, we can obtain the ranking of all alternative as shown in Table III.

It is shown in Table III that the optimal ranking order of these four major cutting machines is $x_2 > x_3 > x_4 > x_1$, and thus the best alternative is x_2, namely, B Firm.

6. Conclusion

TOPSIS method is the one of the classical decision-making methods for solving the MCDM problems with crisp numbers, which has a simple computation process, systematic procedure, and a sound logic that represent the rationale of human choice. In this paper, we have developed the TOPSIS method to deal effectively with the MCDM problems with BPFNs. We have defined distance measure for BPFNs and discussed its properties .We have developed a simple and effective decision method to solve the MCDM problem with BPFNs.

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