

DETERMINATION OF GROUP EXTENSIONS OF AN ABELIAN GROUP

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Abstract- In this paper we will outline the extensions of group, foundations of homology of groups starting with theory of co cycles and second homology group, we will later introduce the technique to compute the possible ways that are put together for any given abelian groups N, K in a group G such that N is a normal subgroup of G and $G/N \cong K$. [3][4]

Keywords- Group extension, Homology of group, Co-cycle, Second homology group, Co-chain, Co-boundary.

1. INTRODUCTION

Consider the two groups N and K . An extension of N by K is a short exact sequence

$$1 \rightarrow N \rightarrow G \rightarrow K \rightarrow 1$$

So if G is an extension of N by K then by identifying N isomorphic to the normal subgroup of G , we have $G/N \cong K$. [3][4][5]

Example- Let $Q = \{ \pm 1, \pm i, \pm j, \pm k \}$ be the quaternion group. Let $H = \langle i \rangle$. Here $o(H) = o(i) = 4$. So $H \cong Z_4$. Also $[Q:H] = 2$. Which means H is normal in Q . So $Q/H \cong Z_2$. So Q is the extension of Z_4 by Z_2 . [5]

Example- The extension of A_n , the Alternating group by Z_2 in the form of short exact sequence are

$$\begin{aligned} 1 \rightarrow A_n \rightarrow A_n \times Z_2 \rightarrow Z_2 \rightarrow 1 \\ 1 \rightarrow A_n \rightarrow D_{2n} \rightarrow Z_2 \rightarrow 1 \end{aligned}$$

Example- The extensions of $Z_n = \langle a \rangle$ by $Z_n = \langle b \rangle$ in the form of short exact sequence are

$$\begin{aligned} 1 \rightarrow Z_n \rightarrow Z_{2n} \rightarrow Z_2 \rightarrow 1 \\ 1 \rightarrow Z_n \rightarrow D_{2n} \rightarrow Z_2 \rightarrow 1 \end{aligned}$$

2. GROUP EXTENSIONS

2.1 EQUIVALENT EXTENSIONS

Definition- The two group extensions of N by K are $1 \rightarrow N \rightarrow G \rightarrow K \rightarrow 1$

$$1 \rightarrow N \rightarrow G' \rightarrow K \rightarrow 1$$

equivalent if there exists a homomorphism $\varphi: G \rightarrow G'$ such that the following diagram commutes.

$$\begin{array}{ccccccc} 1 & \rightarrow & N & \xrightarrow{i} & G & \xrightarrow{\pi} & K \rightarrow 1 \\ & & \downarrow \text{id} & & \downarrow \varphi & & \downarrow \text{id} \\ 1 & \rightarrow & N & \xrightarrow{i'} & G' & \xrightarrow{\pi'} & K \rightarrow 1 \quad [4] \end{array}$$

So by the definition we can see that the two short exact sequences are equivalent

$$\begin{aligned} 1 \rightarrow Z_3 \rightarrow Z_6 \rightarrow Z_2 \rightarrow 1 \\ 1 \rightarrow Z_3 \rightarrow Z_3 \times Z_2 \rightarrow Z_2 \rightarrow 1 \end{aligned}$$

as there is an isomorphism between Z_6 and $Z_3 \times Z_2$ and the following diagram commutes

$$\begin{array}{ccccccc} 1 & \rightarrow & Z_3 & \rightarrow & Z_6 & \rightarrow & Z_2 \rightarrow 1 \\ & & \text{id} \downarrow & & \downarrow \varphi & & \downarrow \text{id} \end{array}$$

$$1 \rightarrow Z_3 \rightarrow Z_3 \times Z_2 \rightarrow Z_2 \rightarrow 1$$

2.2 $C^n(G,M)$

Definition- For $n \geq 1$, consider the abelian group of functions $C^n(G,M)$ of functions from $G^n = G \times G \times G \times \dots \times G$ (n copies) to M under the operation multiplication

$$fg(y_1, y_2, \dots, y_n) = f(y_1, y_2, \dots, y_n)g(y_1, y_2, \dots, y_n) \text{ for } f, g \in C^n(G,M) \text{ and } y_1, y_2, \dots, y_n \in G \text{ and for } n=0, C^0(G,M)=M.$$

The elements of $C^n(G,M)$ are called the n-cochains denoted by d_n . [2][4][7]

2.3 COBOUNDARY HOMOMORPHISM

Definition- For $n \geq 0$, we define the nth coboundary homomorphism from $C^n(G,M)$ to $C^{n+1}(G,M)$ by $d_n(f)(g_1, g_2, \dots, g_n) = \sum_{i=1}^n ((-1)^i f(g_1, g_2, \dots, g_{i-1}, g_i, g_{i+1}, \dots, g_{n+1})) + (-1)^{n+1} f(g_1, g_2, \dots, g_n) + y_1 f(g_2, g_3, \dots, g_n)$. Where $f \in C^n(G,M)$ and $(g_1, g_2, \dots, g_{n+1}) \in C^{n+1}(G,M)$

Here $g_i g_{i+1}$ is on the ith position of f in the group G and d_n are group homomorphism. Also $d_n o d_{n-1} = 0$ for every $n \geq 1$. [2][4]

Definition- $Z^n(G,M) = \text{Ker } d_n, n \geq 0$. The elements of $Z^n(G,M)$ are the n-co cycles.

$B^n(G,M) = \text{Image } d_{n-1}, n \geq 1$. The elements of $B^n(G,M)$ are called n-coboundaries, where $B^0(G,M)=1$.

Here $d_n o d_{n-1} = 0$ for every $n \geq 1$. So $\text{Image } d_{n-1} \subseteq \text{Ker } d_n$. So $B^n(G,M)$ is always a subgroup of $Z^n(G,M)$.

Here $C^{n-1}(G,M) \xrightarrow{d_{n-1}} C^n(G,M) \xrightarrow{d_n} C^{n+1}(G,M)$

Now we define the nth homology group G with coefficients in M as $H^n(G, M) \cong Z^n(G, M) / B^n(G, M)$. [2][4]

2.4 SECOND CHOMOLOGY GROUP

Definition- If we denote the group $Z^2(G, M)$ be the group of 2 co-cycles $G \times G \rightarrow M$ and $B^2(G,M)$ to be group of 2-coboundaries. Then $H^2(G, M) \cong Z^2(G, M) / B^2(G, M)$ is called the second homology group of G with coefficients in M.

Now the theory of extensions arises two cases whenever N is abelian and N is non-abelian. Now we restricted ourselves to the study when N is abelian. If G is an extension of N by K, then we produce a homomorphism $\varphi: K \rightarrow \text{Aut}(N)$ and fix a short exact sequence $1 \rightarrow N \rightarrow G \xrightarrow{\pi} K \rightarrow 1$ and we identify N as a subgroup (Isomorphic image) in E. Here the map $\pi: G \rightarrow K$ is surjective. . So for every $a \in K$ there exist $e_a \in G$ such that $\pi(e_a) = a$. Also there exist a function $l: K \rightarrow G$ such that $\pi o l(a) = \pi(l(a)) = \pi(e_a) = a$ i.e. $\pi o l = \text{identity}$ and $e_a = l(a)$. The element e_a is not uniquely determined by K. Infact the set $\{x: \pi(x) = a\} = \{n e_a: n \in N\}$ where $N = \text{Ker}(\pi)$ Here $e_1 = 1$. [2][3][4][8][6]

Lemma 1- Suppose $e_a e'_a \in G$ both map to a under π . Then $e_a n e_a^{-1} = e'_a n (e'_a)^{-1}$ for every $n \in N$.

Proof- Let $n \in N, \pi(e_a) = \pi(e'_a) = a$

$$\Rightarrow e_a^{-1} e'_a \in \text{Ker}(\pi) = N$$

$$\Rightarrow (e_a^{-1} e'_a) n = n (e_a^{-1} e'_a)$$

$$\Rightarrow e_a n e_a^{-1} = e'_a n (e'_a)^{-1} \text{ for every } n \in N$$

So we have a well defined map $\varphi: K \rightarrow \text{Aut}(N)$ as $\varphi(a) = e_a n e_a^{-1}$. Also notice that $\text{Aut}(N)$ is the subgroup of $\text{Perm}(N)$. So $\varphi: K \rightarrow \text{Perm}(N)$ or we can say that K acts on N as group homomorphism.

Suppose $\varphi: K \rightarrow \text{Aut}(N)$ is an action. We can also write it as the semi-direct product $N \rtimes_{\varphi} K$. Now to define group extension we define $f: K \times K \rightarrow N$ as $f(a, h) = e_a e_h e_{ah}^{-1} = l(a) l(h) l(ah)^{-1}$. Thus we can say that l is a group homomorphism iff $f(a, h) = 1$ for every $a, h \in K$. Here $f(a, h)$ is an element of N. So $\pi(f(a, h)) = \pi(e_a e_h e_{ah}^{-1}) = ah (ah)^{-1} = 1$. Here f is called 2-co cycle. [3]

Lemma 2- Let f be the 2-cocycle and $f(1, a) = f(a, 1) = 1$ for every $a \in K$, further $a, h, k \in K, f(a, h)$

$$f(ah, k) = \varphi_a(f(h, k)) f(a, h)$$

Proof- Since $f(1, a) = e_1 e_a e_a^{-1} = e_a e_a^{-1} = 1$

$$\Rightarrow f(a, 1) = e_a e_1 e_a^{-1} = e_a e_a^{-1} = 1$$

So by associativity of G, $(e_a e_h) e_k = e_a (e_h e_k)$

Since $f(a, h) = e_a e_h e_{ah}^{-1}$

$$\text{so } e_a e_h = f(a, h) e_{ah}$$

Now $(e_a e_h) e_k = (f(a,h) e_{ah}) e_k$
 $= f(a,h) f(ah,k) e_{ahk}$

Also $e_a (e_h e_k) = e_a (f(h,k) e_{hk})$
 $= e_a f(h,k) e_a e_a^{-1} e_{hk}$
 $= \varphi_a (f(h,k)) e_a e_{hk}$
 $= \varphi_a (f(h,k)) f(a,hk) e_{ahk}$

so $f(a,h) f(ah,k) = \varphi_a (f(h,k)) f(a,hk)$ [3]

Lemma 3- Suppose $e'_a = n_a e_a$ for some $n_a \in N$. If $f(a,h) = e'_a e'_h (e_{ah}')^{-1}$
 Then $f(a,h) = (n_a \varphi_a (n_h) n_{ah}^{-1}) f(a,h)$

Proof- Since $f(a,h) = e'_a e'_h (e_{ah}')^{-1}$
 $= (n_a e_a) (n_h e_h) (n_{ah} e_{ah})^{-1}$
 $= (n_a e_a n_h e_h) (e_{ah})^{-1} (n_{ah})^{-1}$
 $= (n_a \varphi_a (n_h) f(a,h) e_{ah}) (e_{ah})^{-1} (n_{ah})^{-1}$
 $= n_a \varphi_a (n_h) f(a,h) (n_{ah})^{-1}$
 $= (n_a \varphi_a (n_h) (n_{ah})^{-1}) f(a,h)$ (since N is abelian) [3]

Lemma 4- Every element of G can be uniquely written as $n e_a$, for some $n \in N$ and $a \in K$. Further $(c e_a) (b e_h) = c \varphi_a (b) f(a,h) e_{ah}$

Proof- Let $x \in G$. Then $\pi(x) = a$
 So $\pi(x e_a^{-1}) = \pi(x) \pi(e_a^{-1}) = \pi(x) (\pi(e_a))^{-1} = a a^{-1} = 1$
 $\Rightarrow x e_a^{-1} \in \ker(\pi) = N$
 $\Rightarrow x e_a^{-1} \in N$
 $\Rightarrow x e_a^{-1} = n$ (say)
 $\Rightarrow x = n e_a$

If $c e_a = b e_h$
 $\Rightarrow \pi(c e_a) = \pi(b e_h)$
 $\Rightarrow a = h$

So by cancelling e_a , we have $c = b$
 which proved the uniqueness

So $(c e_a) (b e_h) = c e_a b e_a e_a^{-1} e_h$
 $= c \varphi_a (b) e_a e_h$
 $= c \varphi_a (b) f(a,h) e_a e_h$ [3]

Lemma 5- For any two groups N and K with a homomorphism $\varphi: K \rightarrow \text{Aut}(N)$ and co cycle $f \in Z^n(K,N)$ relative to φ , we define $G_f = N \rtimes K$ by $(c,a)(b,h) = (c \varphi_a (b) f(a,h), ah)$. Also suppose that we have two equivalent extensions

$$\begin{array}{ccccccc} 1 & \rightarrow & N & \xrightarrow{i} & G & \xrightarrow{\pi} & K \rightarrow 1 \\ & & & & \text{id} \downarrow & \downarrow \varphi & \downarrow \text{id} \\ 1 & \rightarrow & N & \xrightarrow{i'} & G' & \xrightarrow{\pi'} & K \rightarrow 1 \end{array}$$

Then the action $K \rightarrow \text{Aut}(N)$ is same for both extensions and the co cycles of both extensions have same class in second cohomology group [3].

2.5 THEOREM

Let N and K be two groups with N as abelian. If $\varphi: K \rightarrow \text{Aut}(N)$ is a homomorphism, then there is one-one correspondence between the equivalent classes of extensions N by K with action φ and $H^2(K, N)$.

Proof: Consider $1 \rightarrow N \xrightarrow{i} G \xrightarrow{\pi} K \rightarrow 1$ be an extension of N by K . consider the action $\varphi: K \rightarrow \text{Aut}(N)$ and co cycles class in $H^2(K, N)$. Since we know that equivalent extensions have same class in second chomology group. So we can define a map between the equivalent extensions and elements of $H^2(K, N)$. To prove this, we need to show that the map is injective and surjective. If $f \in Z^2(K, N)$ then we produce the extension $1 \rightarrow N \rightarrow G_f \rightarrow K \rightarrow 1$ With action φ and co cycle class is f . Which proves the surjectivity. Now suppose corresponding to the action φ and co cycle class in $H^2(K, N)$ we have

$$1 \rightarrow N \xrightarrow{i} G \xrightarrow{\pi} K \rightarrow 1 \text{ and } 1 \rightarrow N \xrightarrow{i'} G' \xrightarrow{\pi'} K \rightarrow 1$$

Assume that the extensions have co cycle i.e. for co cycle f and $e_a \in G, e'_a \in G', f(a, h) = e_a e_h e_{ah}^{-1}, f'(a, h) = e'_a e'_h (e'_{ah})^{-1}$. Now define $\sigma: E \rightarrow E'$ by $\sigma(ne_a) = ne'_a$. Since we know that every element of G and G' has a unique representation ne_a and ne'_a respectively where $n \in N$ and $a \in G$. So the map is well defined. Also by the lemma 4 the map is group homomorphism and then it is a bijection of uniqueness.

So $\pi'(\sigma(ne_a)) = \pi(ne'_a) = \pi(ne_a)$. Here $\sigma|_N = \text{identity}$, which shows the extensions are equivalent. Also if $H^2(K, N) = 1$, then the semi direct product of N by K , is the only extension up to equivalence. [3][4]

Example- The extension of Z_5 by Z_5 . The action $\varphi: Z_5 \rightarrow \text{Aut}(Z_5) \cong Z_5$ is trivial and $H^2(Z_5, Z_5) \cong Z_{(5,5)} \cong Z_5$. So here are 5 in equivalent extensions of Z_5 by Z_5 . The only groups having cardinality 5^2 are Z_{25} and $Z_5 \times Z_5$. So there is one extension corresponding to the direct product and 4 corresponding to Z_{25} . [3][4]

Example- Extensions of Z_2 by Z_3 with $2|3-1$ is $H^2(Z_2, Z_3) = 1$ with trivial action. So all the extensions will be the semi direct product. [3][4]

3. CONCLUSION

The number of extensions of an abelian group is isomorphic to the second chomology group, so the possible number of extensions of an abelian group are the number of elements in second chomology group. Moreover we can find the extensions of a non abelian group.

4. REFERENCES

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