A Common Fixed Point Result for Three self-mappings in Cone *b*-pentagonal metric spaces

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Abstract: In this paper, we prove a common fixed point result for three self-mappings in cone *b*-pentagonal metric spaces without assuming the normality condition. Our results improve and extend recent known results in the literature.

Keywords: cone b-pentagonal metric space; fixed point; cone metric space.

Introduction and Preliminaries

In 1906, M Frechet [1] introduced the concept of metric space. Banach [2] introduced the concept of Banach contraction mapping principle. Because of these applications of this concept, the study of existence and uniqueness of fixed points of a mapping and common fixed point of one, two or more mappings has become a subject of great interest. The Banach contraction principle in various generalized metric spaces is proved by many authors.

The major generalizations in the fixed point theory are Czerwik's *b*-metric space [15] and Huang and Zhang's [3] cone metric space. These generalizations lead many authors to introduce cone *b*-metric space, Azam et al. [7] introduced the notion of cone rectangular metric space and proved Banach contraction mapping principle in a cone rectangular metric space setting. In 2012, Rashwan and Saleh [8] improve and extended the result of Azam et al. [7] by removing the normality condition. Recently, Garg and Agarwal [9] introduced the notion of cone pentagonal metric space and proved Banach contraction mapping principle in a normal cone pentagonal metric space setting using the normal condition.

Motivated and inspired by these results of [8, 10, 12] it is our purpose in this paper to continue the study of a common fixed point result for three self-mappings in cone b-pentagonal metric spaces setting. Our results improve and extend the results of [7, 8, 9, 10, 12, 16] and many others. We want to extend some well known Banach fixed point theorems which are also valid in cone b-pentagonal metric space. We need the following definitions and results, consistent with [2, 3, 7, 9, 10] in the sequel.

Let E be a real Banach space and let P be a subset of E. By θ we denote the zero element of E and by *int P* the interior of P: The subset P is called a cone if and only if:

1) P is closed, non-empty and $P \neq \{\theta\}$
2) $a, b \in R; a, b \geq \theta, x, y \in P \Rightarrow ax + by \in P,$
3) $P \cap (-P) = \{\theta\}$

Given a cone $P \subseteq E,$ we deine a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P.$ We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y,$ while $x << y$ will stands for $y - x \in \operatorname{int}(P),$ where $\operatorname{int}(P)$ denotes the interior of $P.$

A cone P is called normal if there is a number K > 0 such that for all $x, y \in E,$ the inequality

$$\theta \leq x \leq y \; implies \; \|x\| \leq K \geq \|y.\|$$

The least positive number K satisfies the above is called the normal constant of P. It is well known that $K \geq 1$. In the following, we always suppose that E is a Banach space, P is a cone in E with $\text{int}(P) \neq \theta$ and $\leq$ is a partial ordering with respect to P.

Definition 1.1. *[3]* Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

1) $\theta < d(x,y)$ *for all* $x, y \in X$ *and* $d(x,y) = \theta$ *if and only* if $x = y$;
2) $d(x,y) = d(y,x)$ *for all* $x, y \in X$;
3) $d(x,y) \leq d(x,z) + d(z,y)$ *for all* $x, y, z \in X$.

Then d is called a cone metric on X and (X,d) is called a cone metric space.

Remark 1.1. The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where $E = R$ and $P = [0,\infty)$ P = [0;1) *(e.g., see [3]).*

Definition 1.2. *[7]* Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

1) $\theta < d(x,y)$ *for every* $x, y \in X$ *and* $d(x,y) = \theta$ *if and only* if $x = y$;
2) $d(x,y) = d(y,x)$ *for all* $x, y \in X$;
3) $d(x,y) \leq d(x,w) + d(w,z) + d(z,y)$ *for all* $x, y, z, w \in X$ and for all distinct points $w, z \in X - \{x, y\}$. [Rectangular property].

Then d is called a cone rectangular metric on X and (X,d) is called a cone rectangular metric space.

Remark 1.2. Every cone metric space is cone rectangular metric space. The converse is not necessarily true. *(e.g., see [7]).*

Definition 1.3. *[9]* Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

1) $\theta < d(x,y)$ *for every* $x, y \in X$ *and* $d(x,y) = \theta$ *if and only* if $x = y$;
2) $d(x,y) = d(y,x)$ *for all* $x, y \in X$;
3) $d(x,y) \leq d(x,z) + d(z,w) + d(w,u) + d(u,y)$ *for all* $x, y, z, w, u \in X$ and for all distinct points $u, w, z \in X - \{x, y\}$. [Pentagonal property].

Then d is called a cone pentagonal metric on X and (X,d) is called a cone Pentagonal metric space.

Remark 1.3. Every cone rectangular metric space and so cone metric space is cone pentagonal metric space. The converse is not necessarily true *(e.g., see [9]).*

Definition 1.4. Let X be a nonempty set and let $s \geq 1$ be a given real number. Suppose that the mapping $d : X \times X \to E$ is said to be cone b-metric if and only if for all $x, y, z \in X$ the following conditions are satisfied:

1) $\theta < d(x,y)$ *with* $x \neq y$ *and* $d(x,y) = \theta$ *if and only* if $x = y$;
2) $d(x,y) = d(y,x)$ *for all* $x, y \in X$;
3) $d(x,y) \leq s[d(x,z) + d(z,y)]$.

Then d is a called a cone b-metric on X and (X,d) is called a cone b-metric space.

Definition 1.5. Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that the mapping $d : X \times X \to E$ is said to be cone b-pentagonal metric space then the following conditions hold:

1) $\theta < d(x,y)$ *for every* $x, y \in X$ *and* $d(x,y) = \theta$ *if and only* if $x = y$;
2) $d(x,y) = d(y,x)$ *for all* $x, y \in X$;
3) $d(x,y) \leq s[d(x,z) + d(z,w) + d(w,u) + d(u,y)]$ *for all* $x, y, z, w, u \in X$ and for all distinct points $u, w, z \in X - \{x, y\}$.

Then d is called a cone b-pentagonal metric on X and (X,d) is called a cone b-pentagonal metric space.

Definition 1.6. Let (X,d) be a cone b-pentagonal metric space, $x\in X$ and let $\{x_n\}$ be a sequence in X. Then

1) $\{x_n\}$ converges to x whenever, for every $c\in E$ with $\theta<<c$, there is a natural number N such that $d(x_n,x)<<c$ for all $n\geq N$ We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ $(n\to\infty)$.
2) $\{x_n\}$ be a Cauchy sequence whenever, for every $c\in E$ with $\theta<<c$, there is a natural number N such that $d(x_n,x_m)<<c$ for all $n,m\geq N$.
3) (X,d) is a complete cone b-pentagonal metric space if every Cauchy sequence is convergent.

Definition 1.7. Let P be a cone defined as above and let Φ be the set of non decreasing continuous functions $\varphi: P\to P$ satisfying:

1) $\theta<\varphi(t)<t$ for every $t\in P\setminus\{\theta\}$.
2) the series $\sum_{n\geq 0}\varphi^n(t)$ converge for all $t\in P\setminus\{\theta\}$ From (1), we have $\varphi(\theta)=\theta$ from (2); we have $\lim_{n\to 0}\varphi^n(t)=\theta$ for all $t\in P\setminus\{\theta\}$.

Let f and g be self maps of a nonvoid set X If $w=fx=gx$ for some $x\in X$ then x is called coincidence point of f and g and w is called a point of coincidence of f and g. Also, f and g are said to be weakly compatible if they commute at their coincidence points, that is, $fx=gx \Rightarrow f(g(x))=g(f(x))$.

Lemma 1.4. Let f and g be weakly compatible self mappings of nonvoid set X. If f and g have a unique point of coincidence $w=fx=gx$ then w is the unique common fixed point of f and g.

Lemma 1.5. *[13]* Let (X,d) be a complete cone pentagonal metric space. Let $\{x_n\}$ be a Cauchy sequence in X and suppose that there is natural number N such that:

1) $x_n\neq x_m$ for all $n,m>N$;
2) x_n, x are distinct points in X for all $n>N$;
3) x_n, y are distinct points in X for all $n>N$;
4) $x_n\to x$ and $x_n\to y$ as $n\to\infty$.

Then $x=y$.

Lemma 1.6. *[3]* Let (X,d) be a cone b-metric space. The following properties are often used while dealing with cone b-metric spaces in which the cone is not necessarily normal.

(1) If $u<<v$ and $v\leq w$, then $u<<w$;

(2) If $\theta\leq u<<c$ for each $c\in \operatorname{int} P$ then $u=\theta$;

(3) If $a\leq\lambda a$ where $a\in P$ P and $0<\lambda<1$, then $a=\theta$;

For other basic properties on cone metric space, the authors refer to the paper [3].

2 Main Results

Out of this we arrive at the conclusion that an extension of a common fixed point result for three self-mappings in cone b-pentagonal metric spaces and we give an example to illustrate the result.

Theorem 2.1. Let (X,d) be a complete cone b-pentagonal metric space. Suppose that the mapping $f,g,h: X\to X$ satisfies the following:

$$d(fx,gx)\leq\varphi(d(hx,hy)), \tag{1}$$

for all $x, y \in X$ where $\varphi \in \Phi$. Suppose that $f(x) \cup g(x) \subseteq h(x)$ and $h(x)$ is a complete subspace of X, then the mapping f, g and h have a unique point of coincidence in X. Moreover, if (f, g) and (g, h) are weakly compatible the f, g and h have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Since we can choose $x_1 \in X$ such that $hx_1 = fx_0$ Also we can choose $x_2 \in X$ such that $hx_2 = gx_1$. Continuing this process, having chosen $x_n \in X$ we obtain x_{n+1} such that $hx_{n+1} = fx_n$ and $fx_{n+2} = gx_{n+1}$ for all $n = 0,1,2,........$. If $hx_n = hx_{n+1}$ and $fx_n = gx_n$ and x_n is coincidence point of f, g and h . Hence, we assume that $x_n \neq x_{n+1}$ for all $n \in N$. Then, from (1), it follows that

$$\begin{aligned} d(hx_n, hx_{n+1}) &= d(fx_{n-1}, gx_n) \\ &\leq \varphi[d(hx_{n-1}, hx_n)] \\ &\leq \varphi^2[d(hx_{n-2}, hx_{n-1})] \\ &\quad . \\ &\quad . \\ &\quad . \\ &\quad . \\ &\leq \varphi^n[d(hx_0, hx_1)]. \end{aligned} \tag{2}$$

It again follows that

$$\begin{aligned} d(hx_n, hx_{n+2}) &= d(fx_{n-1}, gx_{n+1}) \\ &\leq \varphi[d(hx_{n-1}, hx_{n+1})] \\ &\leq \varphi^2[d(hx_{n-2}, hx_n)] \\ &\quad . \\ &\quad . \\ &\quad . \\ &\leq \varphi^n[d(hx_0, hx_2)]. \end{aligned} \tag{3}$$

In similar way, it follows that

$$d(hx_n, hx_{n+3}) \leq \varphi^n[d(hx_0, hx_3)] \tag{4}$$

$$d(hx_n, hx_{n+4}) \leq \varphi^n[d(hx_0, hx_4)] \tag{5}$$

Similarly, for $k = 0,1,2,3,........$ it further follows that

$$d(hx_n, hx_{n+3k+1}) \leq \varphi^n[d(hx_0, hx_{3k+1})] \tag{6}$$

$$d(hx_n, hx_{n+3k+2}) \leq \varphi^n[d(hx_0, hx_{3k+2})] \tag{7}$$

$$d(hx_n, hx_{n+3k+3}) \leq \varphi^n[d(hx_0, hx_{3k+3})] \tag{8}$$

By using (2) and cone b-pentagonal property, we have

$$\begin{aligned} d(hx_0, hx_1) &\leq s[d(hx_0, hx_1) + d(hx_1, hx_2) + d(hx_2, hx_3) + d(hx_3, hx_4)] \\ &\leq s[d(hx_0, hx_1) + \varphi(d(hx_1, hx_2)) + \varphi^2(d(hx_2, hx_3)) + \varphi^3(d(hx_3, hx_4))] \\ &\leq s\left[\sum_{i=0}^{3} \varphi^i(d(hx_0, hx_1))\right]. \end{aligned}$$

and

$$d(hx_0, hx_7) \leq s[d(hx_0, hx_1) + d(hx_1, hx_2) + d(hx_2, hx_3) + d(hx_3, hx_4)$$

$$
\begin{aligned}
&+d(hx_4,hx_5)+d(hx_5,hx_6)+d(hx_6,hx_7)]\\
&\le s\left[d(hx_0,hx_1)+\varphi(d(hx_1,hx_2))+\varphi^2(d(hx_2,hx_3))+\varphi^3(d(hx_3,hx_4))\right.\\
&\left.+\varphi^4(d(hx_4,hx_5))+\varphi^5(d(hx_5hx_6))+\varphi^6(d(hx_6,hx_7))\right]\\
&\le s\left[\sum_{i=0}^{6}\varphi^i(d(hx_0,hx_1))\right]
\end{aligned}
$$

Now, by induction, we obtain for each $k=1,2,3,......$

$$
d(hx_0,hx_{3k+1})\le s\left[\sum_{i=0}^{3k}\varphi^i(d(hx_0,hx_1))\right] \tag{9}
$$

Also, using (2), (3), and cone b-pentagonal property, we have that

$$
d(hx_0,hx_5)\le s\left[\sum_{i=0}^{2}\varphi^i(d(hx_0,hx_1))+\varphi^i(d(hx_0,hx_2))\right].
$$

and

$$
d(hx_0,hx_8)\le s\left[\sum_{i=0}^{5}\varphi^i(d(hx_0,hx_1))+\varphi^6(d(hx_0,hx_2))\right].
$$

By induction, we obtain for each $k=1,2,3,......$

$$
d(hx_0,hx_{3k+2})\le s\left[\sum_{i=0}^{3k-1}\varphi^i(d(hx_0,hx_1))+\varphi^{3k}(d(hx_0,hx_2))\right]. \tag{10}
$$

Again, using (2), (4), and cone b - pentagonal property, we have that

$$
d(hx_0,hx_6)\le s\left[\sum_{i=0}^{2}\varphi^i(d(hx_0,hx_1))+\varphi^3(d(hx_0,hx_3))\right].
$$

and

$$
d(hx_0,hx_9)\le s\left[\sum_{i=0}^{5}\varphi^i(d(hx_0,hx_1))+\varphi^{3k}(d(hx_0,hx_3))\right].
$$

By induction, we obtain for each $k=1,2,3,......$

$$
d(hx_0,hx_{3k+3})\le s\left[\sum_{i=0}^{3k-1}\varphi^i(d(hx_0,hx_1))+\varphi^{3k}(d(hx_0,hx_3))\right]. \tag{11}
$$

Using (6) and (9), for $k=1,2,3,......$ we have

$$
\begin{aligned}
d(hx_n,hx_{n+3k+1})&\le\varphi^n\left(s\left[\sum_{i=0}^{3k}\varphi^i(d(hx_0,hx_1))\right]\right)\\
&\le\varphi^n\left(s\left[\sum_{i=0}^{3k}\varphi^i(d(hx_0,hx_1)+d(hx_0,hx_2)+d(hx_0,hx_3))\right]\right)\\
&\le\varphi^n\left(s\left[\sum_{i=0}^{\infty}\varphi^i(d(hx_0,hx_1)+d(hx_0,hx_2)+d(hx_0,hx_3))\right]\right) \qquad (12)
\end{aligned}
$$

Similarly for $k = 1,2,3,......$ and (7) and (10) implies that

$$d(hx_n, hx_{n+3k+2}) \le \varphi^n\left(s\left[\sum_{i=0}^{3k-1}\varphi^i(d(hx_0,hx_1)) + \varphi^{3k}(d(hx_0,hx_2))\right]\right).$$

$$\le \varphi^n\left(s\left[\sum_{i=0}^{\infty}\varphi^i(d(hx_0,hx_1)) + d(hx_0,hx_2) + d(hx_0,hx_3)\right]\right) \tag{13}$$

Again, for $k = 1,2,3,......$ and (8) and (11) implies that

$$d(hx_n, hx_{n+3k+3}) \le \varphi^n\left(s\left[\sum_{i=0}^{\infty}\varphi^i(d(hx_0,hx_1)) + d(hx_0,hx_2) + d(hx_0,hx_3)\right]\right) \tag{14}$$

Thus, by (12), (13), (14), we have, for each m,

$$d(hx_n, hx_{n+m}) \le \varphi^n\left(s\left[\sum_{i=0}^{\infty}\varphi^i(d(hx_0,hx_1)) + d(hx_0,hx_2) + d(hx_0,hx_3)\right]\right) \tag{15}$$

Since $\left(s\left[\sum_{i=0}^{\infty}\varphi^i(d(hx_0,hx_1)) + d(hx_0,hx_2) + d(hx_0,hx_3)\right]\right)$ converges (by definition 1.7), where, $(d(hx_0,hx_1)) + d(hx_0,hx_2) + d(hx_0,hx_3) \in P\setminus\{\theta\}$ and P is closed, then $\left(s\left[\sum_{i=0}^{\infty}\varphi^i(d(hx_0,hx_1)) + d(hx_0,hx_2) + d(hx_0,hx_3)\right]\right) \in P\setminus\{\theta\}$.

Hence

$$\lim_{n\to\infty}\left\{\varphi^n\left(s\left[\sum_{i=0}^{\infty}\varphi^i(d(hx_0,hx_1) + d(hx_0,hx_2) + d(hx_0,hx_3))\right]\right)\right\} = \theta.$$

Then, for given $c >> \theta$ there is a natural number N_1 such that

$$\varphi^n\left(s\left[\sum_{i=0}^{\infty}\varphi^i(d(hx_0,hx_1) + d(hx_0,hx_2) + d(hx_0,hx_3))\right]\right) << c, \text{ for all } n \ge N_1.$$

Thus, from (15) and (16), we have

$$d(hx_n, hx_{n+m}) << c, \text{ for all } n \ge N_1.$$

Therefore, hx_n is a Cauchy sequence in (X,d). Since $h(X)$ is complete subspace of X there exists a point $u, v \in h(X)$ such that $\lim_{n\to\infty} hx_n = v = hu.$

Now, we will show that $hu = fu.$ Given $c >> 0,$ we choose a natural numbers $N_2, N_3 \in N$ such that $d(v, hx_n) << \frac{c}{4s}$ for all $n \in N_2$ and $d(hx_n, hx_{n+1}) << \frac{c}{4s}$ for all $n \in N_3$. Since $x_n \ne x_m$ for $n \ne m$ therefore by b-pentagonal property, we have

$$d(hu, fu) \le s[d(hu, hx_n) + d(hx_n, hx_{n+1}) + d(hx_{n+1}, hx_{n+2}) + d(hx_{n+2}, fu)]$$

$$\leq s[d(v,hx_n)+d(hx_n,hx_{n+1})+d(hx_{n+1},hx_{n+2})+\varphi d(hu,hx_{n+1})]$$
$$< s[d(v,hx_n)+d(hx_n,hx_{n+1})+d(hx_{n+1},hx_{n+2})+\varphi d(v,hx_{n+1})]$$
$$<< s\left[\frac{c}{4s}+\frac{c}{4s}+\frac{c}{4s}+\frac{c}{4s}\right]=c \text{ for all } n\geq N.$$

Where $N:=\max\{N_2,N_3\}$. Since c is arbitarary we have $d(hu,\, fu) << \frac{c}{m}$ $\forall\ m\in N$. Since $\frac{c}{m}\to\theta$ as $m\to\infty$, we conclude $\frac{c}{m}-d(hu,\, fu)\to -d(hu,\, fu)$ as $m\to\infty$. Since P is closed, $-d(hu,\, fu)\in P$.

Hence $d(hu,\, fu)\in P\cap -P$. By definition of cone we get that $d(hu,\, fu)=\theta$, and so $hu=fu$. Hence, v is a coincidence of f,g and h *i.e.,* $hu=gu=fu=v$. Next we show that u is unique. For suppose v' be another fixed point of coincidence of f,g and h such that $fu'=gu'=hu'=v'$, for some $u'\in X$, then,

$$d(v,v')=d(fu,gu')\leq\varphi(d(gu,hu'))=\varphi(d(v,v')).$$

Hence $v=v'$. Since (f,g) and (g,h) are weakly compatible, by Lemma 1.4, v is the unique common fixed point of f,g and h. This completes the proof of the theorem.

Example 2.2. Let $X=\{a,b,c,d,e\}$, $E=R^2$ and $P=(x,y\geq\theta)$ is a cone in E.
Define $d: X\times X\to E$ as follows:

$$d(x,x)=\theta,\ \ \forall\ x\in X;$$
$$d(a,b)=d(b,a)=(4,16);$$
$$d(a,c)=d(c,a)=d(c,d)=d(d,c)=d(b,d)=d(d,b)=(1,4);$$
$$d(a,e)=d(e,a)=d(b,e)=d(e,b)=d(c,e)=d(e,c)=d(d,e)=d(e,d)(5,20).$$

Then (X,d) is a cone d-pentagonal metric space, but (X,d) is not cone rectangular metric space because it lacks the rectangular property:

$$(4,16)=d(a,b)>d(a,c)+d(c,d)+d(d,b)$$
$$=(1,4)+(1,4)+(1,4)$$
$$=(3,12) \text{ as } (4,16)-(3,12)=(1,4)\in P.$$

Define a mapping f,g and $h: X\to X$ as follows:

$$f(x)=a,\quad \forall\ x\in X.$$

and
$$g(x)=\begin{cases} a & if \quad x\neq e \\ b & if \quad x=e\end{cases}$$
$$h(x)=x,\quad \forall\ x\in X.$$

Clearly $f(X)\cup g(X)\subseteq h(X)$ is complete subspace of X. Also, the (f,g) and (g,h) are weakly compatibles. The conditions of Theorem 2.1 holds for all $x,y\in X$, where $\varphi(t)=\frac{1}{9}t$ and a is the unique fixed point of the mappings f,g and h.

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