ON NUMERICAL EVALUATION OF DERIVATIVES OF FRACTIONAL ORDERS

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Abstract : Numerical evaluation of derivatives of fractional order $1-\alpha$, $0<\alpha<1$ has been considered by applying a four point approximation formula meant for derivatives and the technique of numerical evaluation of integrals of fractional orders α **.**

IndexTerms - Fractional derivative, fractional integral, quadrature rule, corrective factors.

MATHEMATICS SUBJECT CLASSIFICATION: 65 D 30

I. INTRODUCTION

In recent years the subject fractional calculus has been found immense and wide applications in different branches of science and engineering, some of which have been highlighted in Oldham and Spanier (2006) and Dalir and Bashour (2010). Due to its immense applications, the numerical treatment of this subject has been drawing more and more attentions. Some of the numerical techniques developed for the evaluation of integrals and derivatives of fractional orders are due to Lether et al. (1982), Diethelm and Walz (1997) and Acharya et al. (2011).

Acharya et al. (2011) have considered the numerical evaluation of integrals of fractional order α , $0<\alpha<1$, denoted as $D^{-\alpha} f(x)$ and given by

$$D^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dx \qquad (1)$$

The definition of integral of fractional order α specified by equation (1) is in the Riemann-Liouville sense.

In this paper we consider the numerical determination of the derivative $D^{1-\alpha} f(x)$ of fractional order $1-\alpha$ which is given by

$$D^{1-\alpha} f(x) = \frac{d}{dx}\left\{ D^{-\alpha} f(x) \right\} \qquad (2)$$

where the function $D^{-\alpha} f(x)$ is differentiable.

II. NUMERICAL APPROXIMATION OF DERIVATIVE OF FRACTIONAL ORDER

It is pertinent to note that the numerical evaluation of the derivative of fractional order $1-\alpha$, involves the integral of fractional order α which should be evaluated numerically as accurately as possible. For an accurate numerical evaluation of the fractional integral $D^{-\alpha} f(x)$, Acharya et al. (2011) have applied *n*-point Gauss-Legendre quadrature rules or Radau *n*-point rules along with corrective factors $C_r(x)$ of order $r \le 5$ which is prescribed as

$$C_r(x) = \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{r} \frac{(-1)^j f^{(j)}(x)}{j!(j+\alpha)} x^{j+\alpha} . \qquad (3)$$

The corrective factor $C_r(x)$ in conjunction with the quadrature rule yields the approximation for $D^{-\alpha} f(x)$ in the following form:

$$D^{-\alpha} f(x) \approx Q_{-\alpha,n}(h;x) + C_r(x) \qquad (4)$$

where $Q_{-\alpha,n}(\cdots)$ is either the *n*-point Gauss-Legendre quadrature rule or the *n*-point Radau rule meant for the numerical approximation of fractional integral $D^{-\alpha}h(x)$ i.e.

$$D^{-\alpha}h(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \frac{h(t)}{(x-t)^{1-\alpha}}\,dt \tag{5}$$

where

$$h(t) = f(t) - \sum_{j=0}^{r}(t-x)^j f^{(j)}(x)/j!\,. \tag{6}$$

For the numerical approximation of the derivative of $D^{-\alpha}f(x)$ we require the numerical approximation of the fractional order integral $D^{-\alpha}f(x)$ at a certain set of points. So it is pertinent that the fractional order integral at the required set of points ought to be found out as accurately as possible. For achieving this the order *r* of the corrective factor $C_r(x)$ and the index *n* of the Gaussian quadrature rule $Q_{-\alpha,n}(\cdots)$ should both be reasonably large such that the accuracy ultimately obtained is acceptable.

So far as the approximation formula for the numerical evaluation of the derivative $\frac{d}{dx}g(x)$ of a differentiable function $g(x)$, is concerned the following 4-point approximation formula is considered:

$$\frac{d}{dx}g(x) \approx \frac{t^3\{g(x+s)-g(x-s)\} - s^3\{g(x+t)-g(x-t)\}}{2st(t^2-s^2)} \tag{7}$$

where the order of accuracy of the formula given by equation (7) is $O(s^2t^2)$ and the positive numbers *s* and *t* are appreciably small and different.

Thus the technique of numerical evaluation of fractional derivative $D^{1-\alpha}f(x)$ of order $1-\alpha$ consists of the following steps.

i. The variable *x* in equation (1) and (4)-(7) is replaced by $x \pm s$ and $x \pm t$ for obtaining the values $D^{-\alpha}f(x \pm s)$ and $D^{-\alpha}f(x \pm t)$ assigning suitable values of *n* (the index of the rule), *r* (the order of the corrective factor) and the parameter *s* and *t*.

ii. The computed results obtained in step (i) are substituted in the approximation formula given by equation (7) which yields the output $D^{1-\alpha}f(x)$ to a desired degree of accuracy.

III. NUMERICAL TESTS AND CONCLUSION

The following derivative of fractional order $\alpha = 1/2$ (semi derivative) is considered for conducting the numerical test.

$$D^{1/2}f(x) = \frac{1}{\Gamma(1/2)}\frac{d}{dx}\left\{\int_0^x \frac{e^t}{(x-t)^{1/2}}\,dt\right\} \tag{8}$$

for which the exact value is $1/\sqrt{\pi x} + e^x\, erf(\sqrt{x})$ [Ref. Oldham and Spanier (2006)]. The parameters *s* and *t* are assigned the values *s*=0.0005 and *t*=0.0008 and the four semi integrals of the function *exp*(*x*) at the four points $x \pm s$ and $x \pm t$ have been found out for $x = 0.4$ and $x = 0.8$. The evaluation of the required semi integrals have been performed by using Gauss-Lengendre and Radau quadrature rules of index *n*=4, 5, 6 and the order of the corrective factor $C_r(x)$ is $r = 8$. The computed results have been appended in Tables 1 and

2. Table-1 depicts the evaluations by the Gauss-Legendre rules and Table-2 depicts the evaluation by the Radau rules which are denoted as $Q_n^{GL}+C_8$ and $Q_n^{R}+C_8$ respectively.

The notation $|Error|$ in the tables indicates $\left|D^{\frac{1}{2}}(e^x)-\text{exact value}\right|$. It is observed from the tables that both the rules have almost the same accuracy. Further if the index *n* of the rule $Q_n(\cdots)$ is raised from 4 to 6, then the accuracy is improved. However, if *n* is relatively less then for maintaining the accuracy the order of the corrective factor i.e. *r* should be large. For $x \le 1$ the values *n*=6 and *r*=8 yield results of appreciable accuracy.

Table-1

Rule	x	$D^{-\frac{1}{2}}f(x\pm s)$	$D^{-\frac{1}{2}}f(x\pm t)$	$D^{\frac{1}{2}}f(x)$	$\|Error\|$
$Q_4^{GL}+C_8$	0.4	0.93913367360970	0.93968289754039	1.83028050143319	4.12×10^{-13}
		0.93730339290374	0.93675444790158		
$Q_4^{GL}+C_8$	0.8	1.76849419627261	1.76921400960899	2.39807803642117	2.15×10^{-10}
		1.76609611712190	1.76537708428257		
$Q_5^{GL}+C_8$	0.4	0.93913367360968	0.93968289754037	1.83028050193223	5.51×10^{-13}
		0.93730339290372	0.93675444790156		
$Q_5^{GL}+C_8$	0.8	1.76849419525529	1.76921400959161	2.39807803620468	6.04×10^{-13}
		1.76609611710479	1.76537708426553		
$Q_6^{GL}+C_8$	0.4	0.93913367360968	0.93968289754037	1.83028050193259	1.86×10^{-13}
		0.93730339290372	0.93675444790156		
$Q_6^{GL}+C_8$	0.8	1.76849419525531	1.76921400959163	2.39807803620532	4.30×10^{-14}
		1.76609611710481	1.76537708426556		

(computations by Gauss-Legendre 4-point, 5-point and 6-point rules)

Table-2

Rule	x	$D^{-\frac{1}{2}}f(x\pm s)$	$D^{-\frac{1}{2}}f(x\pm t)$	$D^{\frac{1}{2}}f(x)$	$\|Error\|$
$Q_4^{R}+C_8$	0.4	0.93913367360944	0.93968289754013	1.83028050192690	5.87×10^{-12}
		0.93730339290349	0.93675444790133		
$Q_4^{R}+C_8$	0.8	1.76849419501617	1.76921400935158	2.39807803319143	3.01×10^{-9}
		1.76609611686869	1.76537708403033		
$Q_5^{R}+C_8$	0.4	0.93913367360968	0.93968289754037	1.83028050193263	1.42×10^{-13}
		0.93730339290372	0.93675444790156		
$Q_5^{R}+C_8$	0.8	1.76849419525502	1.76921400959134	2.39807803620236	2.92×10^{-12}
		1.76609611710452	1.76537708426527		
$Q_6^{R}+C_8$	0.4	0.93913367360968	0.93968289754037	1.83028050193245	3.25×10^{-13}
		0.93730339290372	0.93675444790156		
$Q_6^{R}+C_8$	0.8	1.76849419525531	1.76921400959163	2.39807803620477	5.07×10^{-13}
		1.76609611710481	1.76537708426555		

(computations by Radau 4-point, 5-point and 6-point rules)

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