ON SOME RESULTS OF STARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract : The most prioritized topic while studying univalent analytic function is the Riemann mapping theorem. In this communication, we introduced '**S**' be the class of function **f**in H that are univalent in D. A new subclass of S^*-starlike functions of order α is studied in this paper. Some of the properties of these S^*-starlike function with negative coefficients including the starlikeness, univalence are reflected. Some other aspects such as integral transforms , quasi-hardmard product functions are discussed. Various examples are provided to study the results of negative coefficient functions of order **¼.**

***IndexTerms* - Starlike function, univalent analytic function, quasi-hardmard product.**

I. INTRODUCTION

The concepts of analytic function and univalent are heavily used in mathematics. The function f which is a complex valued by nature is said to be analytic in a domain Ω (a nonempty open connected subset) if it has a uniquely determined derivative at each point of Ω. The function f is defined as univalent in a domain Ω, if it never takes any value more than once, that is, the condition $f(z_1) = f(z_2)$, $z_1, z_2 \in \Omega$ implies $z_1 = z_2$ .A necessary condition for an analytic function f to be univalent in Ω is $f'(z) = 0$ in Ω. This condition is not sufficient which can be seen by considering the function $f(z) = \exp(z)$ whose derivative never vanishes. But clearly it is not univalent in C. The Riemann mapping theorem states that if Ω is a simply connected domain whose boundary consists of more than the point and z_0 is a point in Ω then there exist a unique univalent analytic function f which maps Ω conformally onto the unit disc $D = \{z \in C : |z| < 1\}$, and the properties $f(z_0) = 0$ and $f'(z_0) = 1$.While studying geometric properties of functions univalent and analytic in a simply connected domain Ω with more than one boundary point one may therefore confine, without loss of generality, it is enough to consider functions analytic and univalent in the unit disc D. If the function $f(z) = (\frac{g(z) - g(0)}{g'(0)})$, since $g'(0) \neq 0$ then g is analytic & univalent in D. So, considering f in Das univalent analytic function which satisfies $f(0) = 0$ and $f'(0) = 1$. Let H be the class of functions f analytic in D and normalised by the conditions $f(0) = 0$ and $f'(0) = 1$, and let S be the class of function f in H that are univalent in D. The Taylor series expansion of such a function f about the origin has the form

$$f(z) = z + \sum_{n=2} a_n z^n \qquad (1)$$

Unless otherwise stated explicitly, it is assumed throughout in the sequel that whenever $f \in S$, is in Taylor series representation of the form (1). The koebe function $k(z) = z(1-z)^{-2}$ which maps the unit disc D onto the entire complex plane minus the part of the negative real axis from 1/4 to infinity is theleading example of a function in S. A few illustrative of such functions in S are $z, \frac{z}{(1-z)^{-1}}$ and $\frac{1}{2}\log\frac{1+z}{1-z}$.

In the univalent function theory was initiated by koebe [13] in 1907 on the uniformization of algebraic curves. He discovered that the ranges of all functions in S contain a common disc $|W| \le b$, where b is an absolute constant. The koebe function $k(z) = z(1-z)^{-2}$ shows that $b \le \frac{1}{4}$. Bieberbach's [2] establishes that $b = \frac{1}{4}$. He also proved in the same paper that if $f \in S$ then $|a_2| \le 2$ with equality occurring iff f is a rotation [$g(z) = e^{-i\theta} f(e^{i\theta} z)$] of the koebe function. Motivated by these extremal properties of the koebe function, Bieberabach conjectured that forever

$$F(z) = z + \sum_{n=2} a_n z^n \in S, a_n \le n,\ n = 2, 3, \ldots . \qquad (2)$$

Equality occurs in (2) for each 'n', iff, $f(z)$ is the koebe function $k(z)$ or one of its rotations [$g(z) = e^{-i\theta} f(e^{i\theta} z)$]. Recently, Bieberbach's conjecture has been proved in affirmative [2]. Whenever there is no ambiguity, we will use term univalent functions in the sequel for analytic univalent functions. This section contains some definitions and result concerning the class S and some of the subclasses of S that are needed in the sequel. Bieberbach's inequality $|a_2| \le 2$ has further implication in the geometric theory of conformal mappings. Indeed, any transformation that carries a function of H into another function in H will give some expression for the second coefficient to which we can apply this bound. One important consequence is that the class H there is a limit to the distortion of the boundary, stated by the Koebe distortion theorem. The following results give a basic estimate which leads to the distortion theorem and related results.

Theorem 1.1 *Let* $z = re^{i\theta}(r<1)$. *Then, for every* $f \in S$,

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2r^2}{1-r^2}\right| \le \frac{4r}{1-r^2}. \tag{3}$$

Using (3), the following theorem can be established.

Theorem 1.2 If $f \in S$, then

$$\frac{1-r}{(1+r)^3} \le |f'(z)| \le \frac{1+r}{(1-r)^3} \tag{4}$$

$$\text{and } \frac{r}{(1+r)^2} \le |f'(z)| \le \frac{r}{(1-r)^2} \tag{5}$$

These inequalities are appropriate. Equality occurs at each extreme, iff, f is a suitable rotation of the koebe function. We shall now discuss the properties for functions with positive real part in D. Let's consider P be the class of function $p(z)$is analytic with positive real part in D and$p(0) = 1$. The function $p(z)$ can't notbe univalent. In the study of the class S, the koebe function plays a crucial role.

$$q(z) \equiv \frac{1+z}{1-z} = 1+2\sum_{n=1}^{\infty} z^n \tag{6}$$

In the study of the class Carathedory [3] proved that the coefficients of $p(z)$ satisfying$|p_n| \le 2, n \ge 1$, which is sharp for q(z). Easily we can proved that if p (z) $\in$ P and $z = re^{i\theta}$. then

$$\frac{1-r}{1+r} \le |p(z)| \le \frac{1+r}{1-r} \text{ and } |p'(z)| \le \frac{2}{(1-r)^2}. \tag{7}$$

These inequalities are appropriate. Equality occurs, iff, $p(z) = q(e^{i\alpha}z)$ for some real α. A simple geometric argument in [5] shows that if p(z) $\in$ P, then it satisfies the inequality

$$\left|p(z) - \frac{1+r^2}{1-r^2}\right| \le \frac{2r}{(1-r)^2}. \tag{8}$$

Again equality occurs for $q(e^{i\alpha}z)$ for some real α. Goluzin [5] proved that if p(z) $\in$ P, then

$$\left|\frac{zp'(z)}{p(z)}\right| \le \frac{2r}{(1-r)^2} \tag{9}$$

For which we obtain $\mathrm{Re}\{\frac{zp'(z)}{p(z)}\} \ge -\frac{2r}{(1-r)^2}$. It can also be proved that $|p''(z)| \le \frac{4}{(1-r)^3}$. Further, let $p(\alpha)$ denote the class of analytic functions with $p(0) = 1$ satisfying $\mathrm{Re}\{p(z)\} > \alpha\ (0 \le \alpha < 1)$, $z \in$ D. This class of functions was studied by Liberal and Livingston [9].Closely related to P is the class Q of all functions $f(z) \in H$ whose derivative has a positive real part.We denote the class of convex function by C. An analytic characterization for a function f in H to be convex is due to Robertson [10]. Thus, a function f in H is convex, iff

$$\mathrm{Re}\{z + z\frac{f''(z)}{f'(z)}\} > 0, \quad z \in D. \tag{10}$$

The function $f \in H$ is said to be convex of order $\alpha(0 \le \alpha,1)$ if for $z \in D$.

$$\mathrm{Re}\{1 + \frac{zf''(z)}{f'(z)}\} > 0, \quad z \in D. \tag{11}$$

It is easily seen that $C(0) \equiv C, C(\alpha) \subset C, 0 \le \alpha < 1$ and $C(1) = \{z\}$. If $F(z) = z + \sum_{n=2} a_n z^n$, then it is known that [5] a necessary condition for f to be in C is $|a_n| \le 1$, $n = 2,3, ...$ With equality occurring for the functions $f(z) = z(1-z)^{-1}$. A sufficient condition for f to be in C is that $\sum n^2 |a_n| \le 1$. If $f(z) = \sum_{n=2}^{\infty} a_n z^n$ is in C then Trimble [20] proved that

$$|a_2{}^2 - a_3| \le 1/3(1 - |a_2|). \qquad (12)$$

The bound in (12) is an improvement on the bound of $|a_2{}^2 - a_3|$ for $f \in C$ obtained earlier in [8] and [15]. A domain Ω in the complex plane is said to be starlike with respect to the point $W_0 \in \Omega$ if the line segment joining $W_0 \in \Omega$ to every other point $W \in \Omega$ entirely in Ω.

Definition 1.1 A function $f \in H$ is said to be starlike with respect to the point W_0 if f maps D onto a domain that is starlike with respect to the point W_0. S^* be the class of starlike functions with respect to the origin . It is observed that $C \subseteq S$. The containment is proper since the koebe function $k(z) = z(1-z)^{-2}$ is in S^* but not in C. Robertson [10] proved that a function $f \in H$, iff, $\mathrm{Re}\{\frac{zf'(z)}{f(z)}\} > 0; z \in D.$ (13)

A function $f \in H$ is said to be starlike of order $\alpha(0 \le \alpha < 1)$ if

$$\mathrm{Re}\{\frac{zf'(z)}{f(z)}\} > \alpha; z \in D. \qquad (14)$$

Denote by $S*(\alpha)$, the class of starlike functions of order α. It follows that $S*(0) = S*, S^*(0) \subset S^*, 0 \le \alpha,1$ and $S^*(1) = \{z\}$.The inequality (4) and (14) reveal a close connection between starlike and convex functions. That is a function $f \in C(\alpha)$ iff $zf^*(z) \in S^*(\alpha)$ for $0 \le \alpha < 1$. If $f(z) = z + \sum_{n+2}^{\infty} a_n z^n$ then a necessary condition for f to be in S^* is that [4]

$$|a_n| \le n, n = 2,3... \qquad (15)$$

The inequality (15) is intence for the koebe function $K(z) = z(1-z)^{-2}$. Let $f(z) = z + \sum_{n+2}^{\infty} a_n z^n$ be defined in D. If $\sum_{n=2}^{\infty} |a_n| n \le 1$ then f (z) in S^*. A natural conjecture of starlike leads to the class of spiralike functions, which gives a useful criterion for univalent. A logarithm spiral is a curve. In the complex plane of the form $W = W_0 e^{-\lambda t}$ $(-\infty < t < \infty)$, where W_0 and λ are complex constants with W_0=0 and $\mathrm{Re}(\lambda)$=0. If we take $\lambda = e^{-i\alpha}$ with - ¶/2 < ¶/2, the curve is called α - spiral. For which α (| α | < ¶/2) there is a unique α - spiral which joins a given point W_0=0 to the origin.

A domain D containing the origin is said to be α - spiral like if for each point W_0=0 in D are of the L-spiral from W_0 to the origin lies entirely in D. A function f analytic and univalent in the unit disc, with f(0) = 0 is said to be α -spiral if its range is α -spiral if its range is α spiralike, 0-spiral function are simple the starlike functions. A slight modification of the condition for starlikeness characterize α spiral functions, which is also a sufficient condition for univalence.

Theorem 1.3 Let $f \in H$ and $|\alpha|$ < ¶/2. Then, f is a α spiral in D iff

$$\mathrm{Re}\{e^{i\alpha} \frac{zf'(z)}{f(z)}\} > 0 \; ; z \in D \qquad (16)$$

These functions are introduced by Spaik [8]. Libera [15] introduced the class of α - spiral functions of order $P(0 \le \rho < 1)$, denoted by S(α ,p), by changing the condition (16) to $\mathrm{Re}\{e^{i\alpha} \frac{zf'(z)}{f(z)}\} > p\cos\alpha;$ z $\in$ D. In addition to other results, be found coefficient estimates for such functions thus, it was proved that if $f \in S(\alpha, \mathrm{P}), \mathrm{p}$ then

$$|a_n| \le \left|\sum_{k=0}^{\infty} \frac{2(1-\rho)\cos\alpha e^{-i\alpha} + k}{k+1}\right| \quad \textit{for } n=2,3\ldots \qquad (17)$$

holds, and that equality occurs for the function $f(z) = z(1-z)^{-2(1-\rho)\cos\alpha \times \exp(-i\alpha)}$. It can be noted that S(α,0) is the class of α - spiral functions.

Another incredible subclass ofS in which S^* is the class which is close –to-convex functions. In the unit disc, the function f is analytic by nature and is said to be close-to-convex [19], if there is a convex function ϕ such that

$$\mathrm{Re}\{\frac{f'(z)}{\phi(z)}\} > 0, \quad z \in D. \qquad (18)$$

We denote K be the class of close-to-convex function is univalent, it may be observed that $C \subset S^* \subset KS$. However, a close-to-convex function need not be α-spiral. Let f(z) be defined by (1) and let P(α) be the class of functions of the form f(Z) which satisfy $\mathrm{Re}\{\frac{f(z)}{z}\} > \alpha$ in D and Q(α) be the class of functions f(z) which satisfy $\mathrm{Re}(f'(z)) > \alpha$ for z$\in$ D. We observe that f(z) $\in$Q(α) iff $zf'(z) \in P(\alpha)$ for $0 \le \alpha < 1$.

The univalent analytic functions of class S has been studied widely. The main focus in this areaconcentered on determining the estimates of coefficients and also the estimates for$|f(z)|$and$|f'(z)|$. Failure to settle to Bieberbach's is conjecture. Namely $|a_n| \le n$ for n=2, 3…. Many part of workers attempted to investigate various subclasses of the class of univalent functions. Among such subclasses, 'T' be the class of functions whose non zero coefficients, whose the second on, are negative, that is, an analytic univalent functions f is in T iff it can be expressed in the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n (a_n \ge 0)$$

In 1975,H. Silverman considered a subclasses of T comprised of polynomials having |z|=1 as radius of univalence. For this class, he obtained a necessary and sufficient condition in terms of the co-efficient and with the conformal mapping of univalent functions. According Silverman [16] , co-efficient inequalities, distortion and covering theorems for the subclasses $S*(\alpha)$ and $C*(\alpha)$ of T, the class of starlike functions of order α and the class of functions of order α respectively. Let $P*(\alpha)$ and $Q*(\alpha)$ denote the classes obtained by taking intersection of $P(\alpha)$ and $Q(\alpha)$ with T, that is, $P*(\alpha) = P(\alpha) \cap T$ and $Q^*(\alpha) = Q(\alpha) \cap T$.

II. A SUBCLASSES OF STARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

Let's consider A be the class of function of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (19)$$

which are analytic in the open unit disc $u = \{z : |z| < 1\}$. Further, consider S be the class of all functions in A which are univalent in U. Let $S^*(\alpha), (0 \le \alpha < 1)$ be the subclasses of functions in S which are starlike of orderα .Analytically; $f \in S^*(\alpha)$ iff f is of the form (19) and satisfies $\mathrm{Re}\{\frac{zf'(z)}{f(z)}\} > \alpha$ (z$\in$ U).

Similarly f $\in$ C (α); iff, f is of the form (19) and $\mathrm{Re}\{1 + \frac{zf''(z)}{f'(z)}\} > \alpha$ (z$\in$ U). Now for f $\in$ C(α) we have $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$; For some suitable a_n's and $\mathrm{Re}\{1 + \frac{zf''(z)}{f'(z)}\} > \alpha$ (z$\in$ U). Now, $f'(z) = 1 + \sum_{k=2}^{\infty} k a_k z^{k-1}$ implies $zf'(z) = z + \sum_{k=2}^{\infty} k a_k z^k$.Therefore, z-f is of the form (19). Again, $\mathrm{Re}\{\frac{z(zf')(z)}{zf'(z)}\} = \mathrm{Re}\{1 + \frac{zf''(z)}{f'(z)}\} > \alpha$ (since f is in c (α)). Also, for $zf \in S^*(\alpha)$, we can easily found that f $\in$ C (α). Thus, we have f $\in$ C(α) iff $zf \in S^*(\alpha) \subset S$ and we note that the function f is in S is a starlike function of order α $(0 \le \alpha < 1)$ and type β $(0 < \beta < 1)$, if it satisfies;

$$\left|\frac{\left(\frac{zf'(z)}{f(z)}-1\right)}{\frac{zf'(z)}{f(z)}+(1-2\alpha)}\right|<\beta(z\in U) \qquad (20)$$

Denotes the class of such functions f that are in S and satisfies (20) by S (α,β). S (α,β) is called as the class of starlike functions of order α and β.Further, f $\in$ S is in C (α,β) denotes the class of convex function of order α and β iff f $\in$ S (α,β).We observe that S $(\alpha,1)$ = S $^*(\alpha)$ be the class of starlike functions of order α and S $(0,\beta)$ be a subclass of starlike function according to Padsmanabhan (1968). Further f $\in$ S is in C (α,β), the class of convex function of order α and type β if and only if f $\in$ S (α,β).We denote that S $(\alpha,1)$ + S $^*(\alpha)$, the class of starlike functions of order α and S $(0,\beta)$ is a subclass of starlike function studied by Padsmanabhan (1968). Further f $\in$ S (α,β) : $(0\le\beta<1)$ the values of $\frac{zf'(z)}{f(z)}$ lie in a disk centred at $\frac{1+(1-2\alpha)\beta^2}{1-\beta^2}$ and whose radius is $\frac{2\beta(1-\alpha)}{1-\beta^2}$. Let's consider T be the subclass of S comprised of functions whose non zero coefficients from the second on are negative, that is an analytic and univalent function is in T iff it can be expressed in the form $f(z)=z-\sum_{k=2}^{\infty}a_k z^k$. We denote by T $^*(\alpha)$, C $^*(\alpha)$, S $^*(\alpha,\beta)$, and C $^*(\alpha,\beta)$. The classes obtained by taking intersection of the classes T $^*(\alpha)$, C $^*(\alpha)$, S $^*(\alpha,\beta)$ and C $^*(\alpha,\beta)$, respectively. The classes T $^*(\alpha)$ & C $^*(\alpha)$, and were introduced and studied by Silverman (1975) whereas the classes S $^*(\alpha,\beta)$ and C $^*(\alpha,\beta)$ and were introduced and studied by Gupta and Jain (1976). These classes possess many unique properties and also been derived by Silverman and silvinia (1979), Owa (1983), Kumar (1987) and others.

The objective of this t section is to provide more general class T $^*(\alpha,\beta)$ of analytic univalent functions involving Ruschewegh's derivatives, and then extent some of the results of Silverman (1975), Gupta and Jain (1976) to the class T $^*(\alpha,\beta)$. We also study some other aspects such as quasi-hadamard product and integral transforms of functions in T $^*(\alpha,\beta)$. By proper choices of n, α and β we get the corresponding results for the classes T $^*(\alpha)$, S $^*(\alpha,\beta)$, and its allies classes.

2.1 A general class T $^*(\alpha,\beta)$ of analytic univalent functions

We begin with definition of the class T $^*(\alpha,\beta)$ in this section and deals with the determination of sharp coefficient estimates and comparable results for the class T $^*(\alpha,\beta)$.

Definition 2.1.1 The function f is said to be in class T $^*(\alpha,\beta)$; n=0, 1, 2… if it satisfied the condition.

$$\left|\frac{\frac{z(D^nf(z))'}{D^nf(z)}-1}{\frac{z(D^nf(z))'}{D^nf(z)}+(1-2\alpha)}\right|<\beta(z\in U) \qquad (21)$$

For some $\alpha(\le\alpha<1);\beta(0<\beta\le1)$ and where $D^n f(z)=\left(\frac{z}{(1-z)^{n+1}}\right)^* f(z)$ (here * denotes the convolution of two analytic functions; be the Ruschewegh's derivative of f(z). If $f(z)=z-\sum_{k=2}^{\infty}a_k z^k\,(a_k\ge0)$ then $D^n f(z)=z+\sum_{k=2}^{\infty}\frac{(n+k-1)}{(n!)(k-1)!}a_k z^k$. Taking D we have

$$D^n f(z)=z-\sum_{k=2}^{\infty}\delta(n,k)a_k z^k \;. \qquad (22)$$

Thus $D^0 f(z)$ and $D'f(z)zf'(z)$. Setting n=0 in (21) we have $T^*\circ(\alpha,\beta)$ be the class of functions $f\in T$ satisfying $\left|\frac{\frac{zf'(z)}{f(z)}-1}{\frac{zf'(z)}{f(z)}+(1-2\alpha)}\right|<\beta(z\in U)$. Clearly by (2.1) we have f is in $S^*(\alpha,\beta)$. Thus $T^*\circ(\alpha,\beta)$ reduces to $S^*(\alpha,\beta)$ and similarly $T^*\circ(\alpha,\beta)$ reduces to $C^*\circ(\alpha,\beta)$.We denote $T_n^*(\alpha,1)$ by $T_n^*(\alpha)$ which coin die with the classes $T^*(\alpha)$ and $C^*(\alpha)$ for n=0 and n=1. It will be shown in this section that $T_n^*(\gamma,\beta)\subset T_n^*(\alpha,\beta)\subset T^*(\alpha)$ for and $0\le\alpha<\gamma,1$ and n=0, 1, 2….Now we are going to prove the following coefficient inequalities for functions belonging to the class $T_n^*(\alpha,1)$.

Theorem 2.1.1 Let $f(z)=z-\sum_{k=2}^{\infty}a_k z^k (a_k \geq 0)$. Be analytic in U, then $f \in T_n^*(\alpha,\beta)$, iff

$$\sum_{k=2}^{\infty}\{k(1+\beta)-(1-\beta+2\alpha\beta)\}\times\delta(n,k)a_k \leq 2\beta(1-\alpha) \tag{23}$$

where $\delta(n,k)=\frac{(n+k-1)}{(n!)(k-1)!}$ the result is sharp.

Proof. Let $f(z)=z-\sum_{k=2}^{\infty}a_k z^k (a_k \geq 0)$ is in $T_n^*(\alpha,\beta)$.Therefore for all $Z \in U$ we have

$$\left|\frac{\frac{z(D''f(z))'}{D'f(z)}-1}{\frac{z(D''f(z))'}{D''f(z)}+(1-2\alpha)}\right|<\beta$$

which implies $\left\{\frac{\sum_{k=2}^{\infty}(k-1)\delta(n,k)a_k z^k}{2(1-\alpha)z-\sum_{k=2}^{\infty}(k+1-2\alpha)\delta(n,k)a_k z^k}\right\}<\beta(z\in U)$

Now choose the value of z on the real axis such that $\frac{z(D''f(z))'}{D''f(z)}$ is real .

Therefore $\sum_{k=2}^{\infty}(k-1)\delta(n,k)a_k z^k<\beta 2(1-\alpha)z-\sum_{k=2}^{\infty}(k+1-2\alpha)\delta(n,k)a_k z^k$. Letting $z \to 1$ through positive values we obtain

$$\sum_{k=2}^{\infty}(k-1)\delta(n,k)a_k<\beta 2(1-\alpha)-\sum_{k=2}^{\infty}(k+1-2\alpha)\delta(n,k)a_k$$

Which implies $\sum_{k=2}^{\infty}\{(k+\beta)-(1-\beta+2\alpha\beta)\}\delta(n,k)a_k<2\beta(1-\alpha)$

Conversely, let (23) consists for all admissible values of $\alpha \,\&\, \beta$, the expression can be derived as

$\phi(f)=\left|z(D^n f(z))'-(D^n f(z))\right|-\beta\left|z(D^n f(z))'+(1-2\alpha)(D^n f(z))\right|$

On replacing $D''f(z)$ and $D''f(z)'$ by their Fourier series expansions in the above equations, we have for |z|=r<1,

$$\phi(f)=\sum_{k=2}^{\infty}(k-1)\delta(n,k)a_k z^k-|\,\beta 2(1-\alpha)z-\sum_{k=2}^{\infty}(k+1-2\alpha)\delta(n,k)a_k z^k\,|$$

$$\leq\sum_{k=2}^{\infty}(k-1)\delta(n,k)a_k r^k-\beta\{2(1-\alpha)r-\sum_{k=2}^{\infty}(k+1-2\alpha)\delta(n,k)a_k r^k\}$$

$$=\sum_{k=2}^{\infty}\{k(1+\beta)-(1-\beta+2\alpha\beta)\}\delta(n,k)a_k r^k-2\beta(1-\alpha)r$$

Since the above inequality holds for all r; $0<r<1$, letting $r \to 1$,we get

$$\phi(f)\leq\sum_{k=2}^{\infty}\{k(1+\beta)-(1-\beta+2\alpha\beta)\}\delta(n,k)a_k-2\beta(1-\alpha)\leq 0$$

By (23). Hence in view of (2.1), it follows that $f \in T_n^*(\alpha,\beta)$. The result is unique, the external function being given by

$$f(z)=z-\frac{2\beta(1-\alpha)}{[k(1+\beta)-(1-\beta+2\alpha\beta)]\delta(n,k)}z^k (k\geq 2) \tag{24}$$

Corollary2.1.2 If $f(z)=z-\sum_{k=2}^{\infty}a_k z^k$ is in $T_n^*(\alpha,\beta)$; then $a_k \leq \frac{2\beta(1-\alpha)}{[k(1+\beta)-(1-\beta+2\alpha\beta)]\delta(n,k)}z^k (k\geq 2)$

With equality for each K, for functions $f_k(z)$ given by(24). Setting n=0 and $\beta = 1$ in theorem 2.2.1, we get the following result of Silverman (1978).

Corollary 2.1.3 Let $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ be analytic in U. Then $f \in T^*(\alpha)$ iff

$$\sum_{k=2}^{\infty} (k-\alpha) a_k \le (1-\alpha) \tag{25}$$

Thus result is unique.

Remarks 2.1.4 Putting n=0 in Theorem 2.1.1 and Corollary 2.1.2, we get a result of Gupta and Jain (1976) which in turn leads to Silverman's result (1975) for $\beta = 1$.

Remarks 2.1.5 Since for $0 \le \alpha < \gamma < 1, 0 < \beta \le 1$ and n=0, 1, 2….

$$\frac{k-\alpha}{1-\alpha} < \frac{[k(1+\beta)-(1-\beta)+2\alpha\beta]\delta(n,k)}{2\beta(1-\alpha)} < \frac{[k(1+\beta)-(1-\beta)+2\alpha\beta]\delta(n,k)}{2\beta(1-\gamma)}$$

It follows from Theorem 2.1.1 and Corollary 2.1.3 that $T_n^*(\gamma,\beta)$ ($T_n^*(\alpha,\beta)$ $T^*(\alpha)$) for $0 \le \alpha < \gamma < 1$ and n=0, 1, 2…
Using the co-efficient inequalities in Theorem 2.1.1, we proved the follow comparable results.

Theorem 2.1.6 For n=0, 1, 2…., we haveT ($T_n^*(\alpha,\beta)$) $T_n^*(\gamma,\beta)$ where

$$\gamma = \gamma(\alpha,\beta,n) = \frac{1+3\beta+(n+1)\alpha+(n-1)\alpha\beta}{n+2+(n+4)\beta-2\alpha\beta}.$$

Thus the result is most favourable.

Proof . Suppose $f(z) = z - \sum_{k=2}^{\infty} a_k r^k \in T*_{n+1}(\alpha,\beta)$. Then by theorem 2.1.1 we have

$$\sum_{k=2}^{\infty} \frac{[k(1+\beta)-(1-\beta+2\alpha\beta)]\delta(n+1,k)a_k \le 1}{2\beta(1-\alpha)} \tag{26}$$

To prove that $f \in T_n^*(\gamma,\beta)$, we must show that

$$\sum_{k=2}^{\infty} \frac{[k(1+\beta)-(1-\beta+2\alpha\beta)]\delta(n,k)a_k \le 1}{2\beta(1-\gamma)} \tag{27}$$

Thus (27) will be satisfied if

$$\frac{[k(1+\beta)-(1-\beta+2\gamma\beta)]\delta(n,k)}{1-\gamma} \le \frac{[k(1+\beta)-(1-\beta+2\alpha\beta)](\delta(n+1,k)k \ge 2)}{1-\alpha}. \tag{28}$$

We shall show that the right hand side of (28) is an increasing function of K .This will be true if for n=0, 1, 2…

$$\phi(k) = (n+k+1)(k-1)[(k+1)(1+\beta)-(1-\beta+2\alpha\beta)] - (n+k)k[(k(1+\beta)-(1-\beta+2\alpha\beta)] + 2\beta(n+1)(1-\alpha)$$

is non-negative, which is certainly true as $\phi_n(z) = k(k-1)(1+\beta) \ge 0$ for k≥2. Since the right hand side of (28) is an increasing function of k, putting $k = \gamma$ in (28) we reduce that $\gamma \le \dfrac{1+3\beta+(n+1)\gamma+(n-1)\alpha\beta}{n+\gamma+(n+4)\beta-2\alpha\beta}$.

This proves the theorem.

The result is unique and the external function being given by $f(z) = z - \dfrac{2\beta(1-\alpha)}{(n+2)(1+3\beta-2\alpha\beta)} z^2$. Putting n=0 in Theorem (2.1.6) we get the following inclusion relation which coincides with a result of Silverman (1975) for $\beta = 1$.

Corollary 2.1.7 For $0 \le \alpha < 1$ and $0 < \beta \le 1$ we have $\delta^*(\alpha,\beta) \subset S^*(\gamma,\beta)$, where $\gamma = \gamma(\alpha,\beta) = \dfrac{1+\alpha+3\beta-\alpha\beta}{\gamma(1+2\beta-\alpha\beta)}$. The result is best possible.

2.2 Integral transformation of functions in $T_n^*(\alpha,\beta)$

This particular section narrates that integral transformers of functions in $T_n^*(\alpha,\beta)$ of the type consider by Bernardi (1969).

Theorem 2.2.1 If $f \in T_n^*(\alpha,\beta)$, then the integral transformers

$$F_c(z) = \frac{c+1}{z_c}\int_0^z t^{c-1} f(t)dt \tag{29}$$

are in $t^*(\alpha)$, where $X = X(\alpha,\beta,c,n) = \dfrac{(n+1)(c+2)(1+\beta)+2(2n+nc-c)\beta(1-\alpha)}{(n+1)(c+2)(1+\beta)+2(2n+nc+1)\beta(1-\alpha)}$.

The result is appropriate for the function $f(z) = z - \dfrac{3\beta(1-\alpha)}{(n+1)(1+3\beta-2\alpha\beta)} z^2$.

Proof. Suppose $f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in T*_n(\alpha,\beta)$

$= z - \sum_{k=2}^{\infty} \dfrac{c+1}{c+k} a_k z^k$

In (25) it is sufficient to show that $f(z)$= $\sum_{k=2}^{\infty} \dfrac{(k-\lambda)(c=1)}{(1-\lambda)(c+k)} a_k \le 1$. For $f \in T_n^*(\alpha,\beta)$, we have by Theorem 2.1.1

$$= \sum_{k=2}^{\infty} \frac{[k(1+\beta)-(1-\beta+2\alpha\beta)]\delta(n,k)a_k}{2\beta(1-\alpha)} \le 1$$

Thus, the objective can be satisfied if

$$\frac{(k-\lambda)(c+1)}{(c+k)(1-\lambda)} \le \frac{[k(1+\beta)-(1-\beta+2\alpha\beta)]\delta(n,k)}{2\beta(1-\alpha)} (k \ge 2)$$

or

$$\lambda \le \frac{(c+k)\delta(n,k)[k(1+\beta)-(1-\beta+2\alpha\beta)]-2k(c+1)\beta(1-\alpha)}{(c+k)\delta(n,k)[k(1+\beta)-(1-\beta+2\alpha\beta)]-2\beta(c+1)(1-\alpha)} \tag{30}$$

We shall prove that the right hand side of (30) is an increasing function of k. This will be true if for n=0, 1, 2….

$$\phi_n(k) = (k-1)(c+k+1)\delta(n,k+1)\{(k+1)(1+\beta)-(1-\beta)+2\alpha\beta)\} - k(c+k)\delta(n,k)\{k(1+\beta)-(1-\beta+2\alpha\beta)\} + 2\beta(\delta+1)(1-\alpha) \ge 0 \tag{31}$$

Now for n=0,(31) becomes

$$\phi_n(k) = (k-1)(c+k+1)\{(k+1)(1+\beta)-(1-\beta)+2\alpha\beta)\} - k(c+k)\{k(1+\beta)-(1-\beta+2\alpha\beta)\} + 2\beta(\delta+1)(1-\alpha)$$

=k(k-1)(1+ β) > 0

Theorem 2.2.1 gives that if $f \in T_n^*(0,1)$ then the Libera transform $F_1(z) = \dfrac{2}{z}\int_{\phi_n}^{z} f(t)dt$ is in $T^*(3n+1/3n+2)$

From this it follows that if $f \in T^*(0)$ then $F_1 \in (1/2)$ and $f \in C^*(0)$ implies that $F_1 \in T^*(4/5)$. Also from the recursive formula

$$(n+1)\phi_{n+1}(k)(n+k)\phi_n(k) + (k-1)(c+k+1)\delta(n,k+1) \{(k+1)(1+\beta)-(1-\beta+2\alpha\beta)\} - 2(k-1)(c+1)\beta(1-\alpha); n = 0,1,2.... \tag{32}$$

we obtain

$$(n+k)\phi_n(k) \le (n+1)\phi_n(k); n = 0,1,2... \tag{33}$$

From (32) and (33), (31) follows. Hence the right hand side of (30) is an increasing function by putting k=2 in (30) we get,

$$X \le \frac{(n+1)(c+2)(1+\beta)+2(2n+nc-c)\beta(1-\alpha)}{(n+1)(c+2)(1+\beta)+2(2n+nc+1)\beta(1-\alpha)}$$

This complete the proof. Setting n=0 in Theorem 2.2.1,we get the following result.

Corollary 2.2.2 If $f \in S^*(\alpha,\beta)$; then the integral transformers $F_c(z)$ defined by (2.3.1) are in $T^*(\lambda_1)$;

where, $$\lambda_1 = \lambda_1(\alpha,\beta,c) = \frac{(c+2)(1+\beta)-2c\beta(1-\alpha)}{(c+2)(1+\beta)-2\beta(1-\alpha)}.$$

Taking n=1 in Theorem 2.2.1, we have

Corollary 2.2.3 If $f \in C^*(\alpha,\beta)$; then the integral transformers $F_c(z)$ defined by (2.3.1) are in $T^*(\lambda_2)$;

Where $$\lambda_2 = \lambda_2(\alpha,\beta,c) = \frac{(c+2)(1+\beta)-2\beta(1-\alpha)}{(c+2)(1+\beta)+\beta(1-\alpha)(c+3)}.$$

Remarks 2.2.4 It is interesting to note that for c=1 and (α,β) = (0, 1).

2.3 Quasi – Hadamard product for functions in $T_n^*(\alpha,\beta)$

Let $f(z) = z - \sum_{k=2}^{\infty} a_k z^k, a_k \ge 0$. Then their quasi – Hadamard product (f*g) (z) is defined by (Kumar 1987, Owa 1983)

$$(f * g)(z) = f(z) * g(z) = z - a_k b_k z^k$$

The following results for functions can be shown in $T_n^*(\alpha,\beta)$.

Theorem 2.3.1 If $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z - \sum_{k=2}^{\infty} b_k z^k$ are in $T_n^*(\alpha,\beta)$, then the quasi-Hadamard product (f*g)(z) = $z - \sum_{k=2}^{\infty} a_k b_k z^k \in T_n^*(\gamma,\beta)$, where

$$\gamma = \gamma(\alpha,\beta,n) = \frac{(n+1)^2(1+3\beta-2\alpha\beta)^2 - 2(n\lambda+2)\beta(1+3\beta)(1-\alpha)^2}{(n+1)^2(1+3\beta-2\alpha\beta)^2 - 4(n+2)\beta^2(1-\alpha)^2} \quad (34)$$

The best possible result for the functions $f(z) = g(z) = z - \frac{2\beta(1-\alpha)}{(n+2)(1+3\beta-2\alpha\beta)} z^2$.

Proof. Suppose f(z) and g(z) are in $T_n^*(\alpha,\beta)$. By taking reference to Theorem 2.1.1 we have

$$\sum_{k=2}^{\infty} \frac{[k(1+\beta)-(1-\beta+2\alpha\beta)]\delta(n,k)a_k \le 1}{2\beta(1-\alpha)} \quad (35)$$

And $$\sum_{k=2}^{\infty} \frac{[k(1+\beta)-(1-\beta+2\alpha\beta)]\delta(n,k)a_k \le 1}{2\beta(1-\gamma)}. \quad (36)$$

Since f(z) and g(z) are analytic in U, So is f(z)*g(z).Further,for γ defined as in (34), and (35) and (36) we have

$$\le \sum_{k=2}^{\infty} \frac{[k(1+\beta)-(1-\beta+2\alpha\beta)]\delta(n+1,k)^2}{2\beta(1-\alpha)} a_k b_k$$

$$\le \sum_{k=2}^{\infty} \frac{[k(1+\beta)-(1-\beta+2\alpha\beta)]\delta(n+1,k)^2}{2\beta(1-\alpha)} a_k.$$

Therefore, by Theorem 2.1.1 f(z)*g(z) belongs to $T^*{}_{n(n+1)}(\gamma,\beta)$, where γ is given by (34). This completes the proof of the theorem.

Putting n=0 and β=1 in the previous expand theorem, we deduce the following result.

Corollary 2.3.2 If $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z - \sum_{k=2}^{\infty} b_k z^k$ are elements of $T^*(\alpha)$,thenh(z)=f(z)*g(z) $= z - \sum_{k=2}^{\infty} a_k b_k z^k$ is an element of $C^*(\gamma)$, where k=2 **and** $\gamma = \frac{(4-3\alpha)}{(2-\alpha)^2}$.

Remarks 2.3.3 From a result of Kumar (1987,p.73,Theorem A), We note that if f(z) and g(z) are in $T_n^*(\alpha)$, then (f*g)(z) is in $C^*(\alpha)$. However, Corollary 2.3.2 shows that if f(z) and g(z) are in $T^*(\alpha)$ then (f*g)(z) belongs to $C^*(4\alpha-3\alpha/2-\alpha)^2$ since $\frac{(4-3\alpha)\alpha}{2-\alpha^2}\in c(\alpha)$.Thus, we observe that the class to which the quasi-Hadmard product belongs, determined by us is much smaller than that obtained by Kumar (1987).From this follows that our result is more inclusive and applicable and thus improves the result of Owa (1983) and Kumar (1987).

Theorem 2.3.4 If $f\in T_n^*(\alpha,\beta);$ and $g\in T_{n+1}^*(\alpha,\beta);$ then $f^*g\in T_{n+1}^*(\alpha,\beta);$ where

$$\gamma=\gamma(\alpha,\beta,n)=\frac{(n+1)(1+3\beta-2\alpha\beta)^2-2\beta(1+3\beta)(1-\alpha)^2}{(n+1)(1+3\beta-2\alpha\beta)^2-4\beta^2(1-\alpha)^2}$$

Thus result is possible for the functions

$$f(z)=z-\frac{2\beta(1-\alpha)}{(n+1)(1+3\beta-2\alpha\beta)}z^2 \text{ and } g(z)=z-\frac{2\beta(1-\alpha)}{(n+2)(1+3\beta-2\alpha\beta)}z^2.$$

Theorem 2.3.5 If $f(z)=z-\sum_{k=2}^{\infty}a_kz^k$ and $g(z)=z-\sum_{k=2}^{\infty}b_kz^k$; $0\le b_k\le 1, k=2,3\ldots$ then $f^*g\in T_n^*(\alpha,\beta).$

Proof. Since $\sum_{k=2}^{\infty}[k(1+\beta)-(1-\beta-2\alpha\beta)]\delta(n,k)a_kb_k\le\sum_{k=2}^{\infty}[\frac{[k(1+\beta)-(1-\beta-2\alpha\beta)]\delta(n,k)a_k}{2\beta(1-\alpha)}\le\infty.$ From Theorem 2.1.1,it follows that $f^*g\in T_n^*(\alpha,\beta).$

Remarks 2.3.6 The function g(z) can't be univalent in the above theorem. For instance, if $g(z)=z-\frac{a}{a+b}z^2$ where a<b<a, then $\frac{a}{a+b}$<1 but $g'(z)=1-\frac{2a}{a+b}z=0$ for $z=\frac{a+b}{2a}$ which lies inside U. Hence g(z) is not univalent.

III. CONCLUSION

In this paper, we introduce "S" be the class of function f in H that are univalent in D. We obtain a new subclass of S^* starlike functions of order α. Further, several properties of the S^*-starlike function with negative coefficients including the starlikeness and univalence are studied. Finally, various other aspects such as integral transforms and quasi-hardmard product functions are presented.

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