Second Order MHD Flow and Heat Transfer Over an Exponentially Stretching Sheet With Viscous and Ohmic Dissipations

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Abstract

Momentum boundary layer of viscoelastic second order fluid arising due to exponentially stretching of boundary sheet, taking into account the effect of uniform transverse magnetic field and normal electric field, has been considered for analysis. Thermal boundary layer formed over the non-isothermal boundary wall, taking into account the viscous dissipation and Ohmic dissipation due to transverse magnetic field and electric field has been investigated. Highly non-linear momentum boundary layer equation and thermal boundary layer equation are converted into similarity equations and then solved numerically by employing fifth order Runge-Kutta-Fehlberg method with shooting. The results are analysed for the situation when stretching boundary sheet is prescribed by non-isothermal temperature and variable heat flux, varying exponentially with the flow directional coordinate x. The effects of various physical parameters like viscoelastic parameter, Prandtl number, local Reynolds number, Eckert number Hartmann number and applied electric parameter on various momentum and heat transfers characteristics are analysed. Some of the important findings of this paper are.

(i) Reduction of velocity with the increase of local normal electric field is more in case of viscous fluid flow compared to viscoelastic fluid flow.

(ii) The skin-friction coefficient C_f would decrease with the increase of the values of local Reynolds number Re_x and increase with the increase of the values of local normal electric parameter.

(iii) The combined effect of increasing values of Prandtl number Pr and Hartmann number Mn is to increase temperature near stretching sheet largely in presence of local normal electric field.

(iv) When the local normal electric field is present then heat transfer takes place from the adjacent fluid layer to the boundary wall with faster rate.

Keywords:

Viscoelastic fluid, skin-friction, heat transfer, exponential stretching, viscous dissipation and Ohmic dissipation.

1. Introduction

Initiating with the pioneering works of SAKIADIS there was an extensive theoretical study over last few decades on the boundary layer flow problem over a stretching sheet as qualitative analysis of this study has bearing on industrial applications such as polymer sheet or filament extrusion from a dye or long thread between feed roll or wind-up roll, glass fiber and paper production, drawing of plastic films, liquid films in condensation process etc. Taking consideration of the fact that most of the fluids associated in industrial applications are more viscoelastic in nature than viscous the studies in the subsequent years gained momentum on various aspects of momentum and heat transfer characteristics in a viscoelastic boundary layer second order fluid flow over stretching sheets (RAJAGOPAL et al., DANDAPAT et al., CORTELL VARJRAVELU and ROLLINS and MAHAPATRA and GUPTA, POP and SOUNDALGEKAR, ABEL et al. Recent work of DANDAPAT et al. has shown that the magnetic field, one of the controlling forces, has stabilizing effect on the boundary layer flow. Keeping in mind some specific industrial applications such as in polymer processing technology, some attempts have been made to investigate theoretically the effect of transverse magnetic field on boundary layer flow characteristics (ANDRESSON, CHAR and LAWRENCE and RAO). ANDRESSON presented a systematic mathematical analysis on MHD flow of second order viscoelastic fluid over an impermeable stretching sheet and showed that magnetic parameter has same effect as that of viscoelastic parameter in flow characteristics. LAWRENCE and RAO discussed the non-uniqueness of the MHD flow of viscoelastic fluid and discussed some theoretical aspects of the solution of the momentum boundary layer equation. However, all these studies are concerned with flow characteristics only associated with the viscoelastic fluid flow over linearly stretching sheet.

Another important phenomenon heat transfer plays an important role in industrial and technological applications as the rate of cooling influences a lot to the quality of the final product with desired characteristics. In view of this some researchers have presented works on MHD flow and heat transfer in a electrically conducting viscoelastic boundary layer flow over a linear stretching sheet (DANDAPAT and GUPTA, ROLLINS and VAJRAVELU, LAWRENCE and RAO , CORTELL , CHAR and SONTH et al). However, in all these analysis researchers have neglected the important controlling force, the electric field, which could effect the momentum and heat transfer characteristics in a viscoelastic boundary layer fluid flow. Vajravelu and Rollins presented a mathematical analysis on heat transfer in an electrically conducting fluid over a stretching surface taking into account the magnetic field only. This study deals with steady state aspects of all the physical quantities. Subsequently, CHAMKHA presented an analysis on unsteady state hydromagnetic flow and heat transfer from a non-isothermal stretching sheet in a porous medium. HELMY presented a work on MHD unsteady free convection flow past a vertical porous plate. The electric field has been excluded from all these studies. Moreover, their analyses are confined to the viscous fluid only. Interestingly, CHIAM presented heat transfer analysis taking into consideration the variable thermal conductivity for slightly different kind of problem of stagnation-point flow towards a stretching sheet. However, this study is also concerned to viscous fluid and without any kind of dissipation and source/sink effect. Recently ZAKARIA presented a work on unsteady state magnetohydrodynamic viscoelastic boundary layer flow and heat transfer over a stretching sheet taking consideration of induced magnetic field. However this work neglected the Ohmic dissipation due to applied magnetic as well as electric field. More recent work of MUKHOPADHYAY and DANDAPAT presented a different kind of study on stability analysis of viscous liquid film flowing down an inclined plane in presence of a uniform normal electric field. This analysis is confined to viscous fluid flow only. Moreover, all the above mentioned Heat transfer anlysis have been carried in the boundary layer formed by linearly stretching sheet. However, GUPTA and GUPTA have pointed out that realistically stretching of the sheet might not necessarily be linear. This situation was dealt by KUMARAN and RAMANAIAH [1996] in their work on boundary layer flow over a quadratic stretching sheet. Recently Ali investigated thermal boundary layer by considering a power law stretching surface. A new dimension is added to this investigation by ELBASHBESHY who examined the flow and heat transfer characteristics by considering exponentially stretching sheet. However, all the above mentioned existing works on non-linear stretching sheet are confined to the viscous fluid flow in absence of magnetic field.

Hence, keeping all the above mentioned works in mind present authors envisage to investigate two dimensional steady state incompressible viscoelastic boundary layer flow over an exponentially continuous stretching sheet with uniform transverse magnetic field and applied normal electric field. Highly non-linear momentum equation and variable coefficient non-homogeneous heat transfer equation are solved numerically by employing fifth order Runge-Kutta-Fehlberg method with shooting and Newton's iteration. We confine our analysis on the effect of various physical parameters like viscoelastic parameter, Prandtl number, local Reynolds number, Eckert number, Hartmann number and local normal electric parameter on various momentum and heat transfer characteristics for two general types of non-isothermal boundary heating process, namely (i) prescribed surface temperature (PST) (ii) prescribed boundary heat flux (PHF), varying quadratically with the flow directional coordinate.

Nomenclature

x flow directional coordinate

y coordinate along normal to the stretching sheet

u, v velocity components

k thermal conductivity

k_0 elastic parameter

μ dynamic viscosity at small rate of shear

U_0 characteristic velocity

l characteristic length

k_1^* viscoelastic parameter

η similarity variable

C_f local skin-friction coefficient

Re_x local Reynolds number

Pr Prandtl number

E Eckert number

E_1 local normal electric parameter

Mn	Hartmann number
f	stream function
θ	non-dimensional temperature parameter in PST case
α_1, α_2	are normal stress moduli

2.Formulation of the Problem

Governing Equations

Consider two-dimensional incompressible steady state electrically conducting viscoelastic fluid flow over a continues stretching sheet. In the present study the flow is considered to be generated by stretching of an elastic boundary sheet issued from a slit with the application of two equal and opposite forces in such way that velocity of the boundary sheet is of exponentially flow directional coordinate x (Fig.1). We take into account the frictional heating due to viscous dissipation as the fluid considered for analysis is of viscoelastic type which possesses viscous property too. The flow region is exposed under uniform transverse magnetic fields $\vec{B} = (0, B_0, 0)$ and uniform normal electric field $\vec{E} = (0,0, E_0)$,. Such application of electric field and magnetic field stabilizes the boundary layer flow (DANDAPAT and MUKHOPADHYAY . The electric field $\vec{E} = (0,0, E_0)$, and magnetic field $\vec{B}_0 = (0, B_0, 0)$ satisfy the following MHD equations:

Maxwell's equations

$$\nabla.\vec{B} = 0 \text{ and } \nabla \times \vec{E} = 0, \qquad (2.1)$$

When magnetic field is not so strong then electric field and magnetic field obey **Ohm's law.**

Ohm's law

$$\vec{J} = \sigma\left(\vec{E} + \vec{q} \times \vec{B}\right) \qquad (2.2)$$

Here J is the Joule current.

We assume that magnetic Reynolds number of the boundary layer fluid flow is small so that induced magnetic field, induced electric field and Hall effect may be neglected. This assumption is quite realistic as flow is caused only by stretching of the boundary. We also consider the electric field as a result of polarization of charges to be negligible. The presence of chemically inactive diffusive species in the boundary layer is assumed to be low and hence Sorest–Dufour effects are negligible. Further, we assume the fluid is more viscous in nature than elastic and so we neglect elastic deformation effects. Under the above stated physical situation, the governing boundary layer equations for momentum and heat transfers are the modified version of the equations of ANDERSSON , KHAN and SANJAYANAND and CORTELL and those are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (2.3)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\gamma\frac{\partial^2 u}{\partial y^2}-k_0\left\{u\frac{\partial^3 u}{\partial x\partial y^2}+v\frac{\partial^3 u}{\partial y^3}-\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}\right\}-\frac{\sigma}{\rho}\left(E_0B_0+B_0^2u\right) \qquad (2.4)$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2}+\frac{\mu}{\rho c_p}\left(\frac{\partial u}{\partial y}\right)^2+\frac{\left(u B_0+E_0\right)^2\sigma}{\rho c_p} \qquad (2.5)$$

Momentum boundary layer equation (2.4) takes into account the Lorentz force $\frac{\vec{J}\times\vec{B}}{\rho}$ due to electo-magnetic effect. In deriving the above momentum boundary layer equation (2.4) it is assumed that the normal stress is of the same order of magnitude as that of the shear stress, in addition to usual boundary layer approximations. Thermal boundary layer equation (2.5) takes into account the Joule heating or Ohmic dissipation due to the magnetic as well as electric fields. Viscous dissipation is accounted in the heat transfer analysis by the term $\frac{\mu}{\rho c_p}\left(\frac{\partial u}{\partial y}\right)^2$ with the assumption that viscoelastic fluid is more viscous in nature than elastic. With this assumption we neglect elastic deformation in comparison with the viscous dissipation. Moreover, model equation (2.4) has been derived on the assumption that the viscoelastic parameter k_1^* is very small at low shear rate (RAJAGOPAL et al.).

3. Boundary conditions on velocity

The flow is assumed to be generated solely by stretching the surface with a horizontal velocity which is exponential function of flow directional coordinate x. Hence we employ the following boundary conditions.

$$u=U_w(x)=U_0\exp\left(\frac{x}{l}\right),\ v=0 \quad at \quad y=0$$
$$u=0 \qquad as \ \ y\to\infty. \qquad (3.1)$$

The Eq. (2.4) is a third order partial differential equation of the dependent variable u. However, the number of prescribed boundary conditions on u given by the equation (3.1) are two, which is one less than number required to obtain unique solution of the equation. In this regard let us glimpse of literature of boundary conditions of such problems. Literature review reveals that in most of the boundary layer flow problems of viscoelastic fluids over solid surfaces this mismatching exists and that leads to the existence of non-unique solutions (RAJAGOPAl et al. , TROY et al. , CHANG, DANDAPAT and GUPTA , LAWRENCE and RAO , CORTELL, RAPTIS and PERDIKIS). Most of the researchers suggested reasonably to consider the particular solution presented by TROY et al. involving the exponential term as realistic one for the case of viscoelastic fluid flow. The numerical solutions of RAJAGOPAL et al. (and) assumed the solution as series expansion of the dependent quantity (stream function $f(\eta)$) up to first order of viscoelastic parameter. LAWRENCE and RAO gave fairly reasonable arguments in support of such expansion of the solution by pointing out that the boundary layer equation was valid only for small values of k_1^*. Therefore, the solution would be slight perturbation from viscous fluid flow solution to take into account the first order viscoelastic term. Assumption of such expansion of solution would reduce the order of differential equation by one and make the order of the governing boundary layer equation of stream function equal to the number of prescribed boundary conditions (MAHAPATRA and GUPTA).

Solution of the momentum boundary layer equation

To solve the boundary layer equation (2.4), we make use of the following stream function, which satisfy the continuity equation.

$$u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x} \qquad (3.2)$$

Stream function $\psi(x, y)$ is defined as follows

$$\psi(x, y) = \sqrt{2\gamma l U_0}\, f(\eta)\, Exp\left(\frac{x}{2l}\right)$$

$$\eta = y\sqrt{\frac{U_0}{2\gamma l}}\, Exp\left(\frac{x}{2l}\right) \qquad (3.3)$$

Here $f(\eta)$ is the dimensionless stream function and η is the pseudo-similarity variable. Making use of the equation (3.3) in the equation (2.4) we obtain a fourth order non-linear quasi-ordinary differential equation of the following form.

$$f_\eta^2 - ff_{\eta\eta} = f_{\eta\eta\eta} - k_1^*\left[3f_\eta f_{\eta\eta\eta} - \frac{1}{2}ff_{\eta\eta\eta\eta} - \frac{3}{2}f_{\eta\eta}^2\right] - 2Mn^2\left(E_1 + f_\eta\right) \qquad (3.4)$$

Here $k_1^* = \frac{k_0 U_w}{\gamma\, l}$ is the dimensionless local viscoelastic parameter, $Mn = \sqrt{\frac{\sigma}{\rho b}}B_0$ is Hartmann number, $E_1 = \frac{E_0}{B_0 U_w}$ is the local normal electric parameter and the subscript η stands for differentiation with respect to η.

The boundary conditions on stream function f of the equation (3.1) in view of the transformations take the following non-dimensional form.

$$f(0) = 0, \quad f_\eta(0) = 1, \quad f_\eta(\infty) = 0 \qquad (3.5)$$

Here we notice that the presence of elasticity in the fluid gives rise to the forth-order differential equation (3.4). However, we have been prescribed only three boundary conditions given by equation (3.5). So, in order to obtain unique solution of the equation (3.4) we are supposed to impose one more boundary condition. We overcome this insufficiency of boundary conditions following the method of solution presented by Rajagopal et al. and MAHAPATRA and GUPTA . They reasonably argued that in case of boundary layer flow of viscoelastic fluid with short memory, the characteristic time scale associated with the motion is large compared with the relaxation time of the fluid. Thus terms of order k^2, k^3 and higher orders may be neglected and therefore we may seek the solution of the equation (3.4) in the form

$$f = f_0(\eta) + k_1^* f_1(\eta) + 0\left(k_1^{*2}\right) \qquad (3.6)$$

Simplifying the equation (3.4) by using the series expansion of the equation (3.6) and equating the constant terms and the coefficient of k_1^* to zero we deduce the following equations for $f_0(\eta)$ and $f_1(\eta)$.

$$2f_{0\eta}^2 - f_0 f_{0\eta\eta} = f_{0\eta\eta\eta} - 2Mn^2\left(E_1 + f_{0\eta}\right) \qquad (3.7)$$

$$f_{1\eta\eta\eta} - 2Mn^2 f_{1\eta} - 4f_{0\eta}f_{1\eta} + f_0 f_{1\eta\eta} + f_1 f_{0\eta\eta} = 3f_{0\eta}f_{0\eta\eta\eta} - \frac{1}{2}f_0 f_{0\eta\eta\eta\eta} - \frac{3}{2}f_{0\eta\eta}^2 \qquad (3.8)$$

Substituting the series expansion of the equation (3.6) in the boundary conditions equation (3.5) we obtain boundary conditions for $f_0(\eta)$ and $f_1(\eta)$ in the following form.

$$f_0(0) = 0, \quad f_{0\eta}(0) = 1, \; f_{0\eta}(\infty) = 0 \qquad (3.9)$$

$$f_1(0) = 0, \quad f_{1\eta}(0) = 0, \; f_{1\eta}(\infty) = 0 \qquad (3.10)$$

Here we notice that zeroth order stream function equation (3.7) as a third order equation of $f_0(\eta)$ for which three boundary conditions are prescribed by equation (3.9). The first order stream function equation (3.8) is also a third order equation of $f_1(\eta)$ for which three boundary conditions are prescribed by equation (3.10). Since the order of the differential equations (3.7) and (3.8) matches well with the number of boundary conditions prescribed by the equations (3.9) and (3.10) respectively, the equations (3.7) and (3.8) would produce unique solutions.

It is to be mentioned that expansion of $f(\eta)$ in a series (3.6) does not constitute a singular perturbation expansion as no boundary condition is ignored in obtaining two well posed boundary values problem by equations (3.7) and (3.9) and equations (3.8) and (3.10). In fact, in a single perturbation problem, apart from expansion of the form (3.6) an asymptotic expansion is also needed and then matching of the two expansion is done. No such things are being done in the procedure adopted in this paper.

The dimensionless local skin-friction coefficient C_f is expressed as

$$C_f = -\frac{\left(\gamma\frac{\partial u}{\partial y} - k_0\left\{u\frac{\partial^2 u}{\partial x\partial y} + v\frac{\partial^2 u}{\partial y^2} - 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right\}\right)}{(bx)^2} \; at \; y=0 \qquad (3.11)$$

$$= -\frac{1}{\sqrt{\mathrm{Re}_x}} f_{\eta\eta}^{(0)}\left(1-\frac{7}{2}k_1^*\right)$$

where $\mathrm{Re}_x = \frac{U_w l}{\gamma}$ is the local Reynolds number and $f_{\eta\eta}^{(0)} = f_{0\eta\eta}^{(0)} + k_1^* f_{1\eta\eta}^{(0)}$.

4.Similarity Solution of the heat transfer equation

Case A: prescribed surface temperature (PST)

In order to solve the thermal boundary layer equation (2.5) in PST case we consider non-isothermal temperature boundary condition as follows.

$$\begin{aligned} &T = T_w = T_\infty + T_0 Exp\left(\frac{v_0 x}{2l}\right) && at \quad y=0 \\ &T \to T_\infty && as \quad y \to \infty \end{aligned} \qquad (4.1)$$

where A_0 is the parameter of temperature distribution on the stretching surface, T_w stands for stretching sheet temperature and T_∞ is the temperature far away from the stretching sheet. We have considered the above form of quadratic power law temperature boundary condition on stretching sheet in order to obtain local similar solution of the equation (2.7).

Keeping in mind to obtain similarity equation from the thermal boundary layer equation (2.7) we define dimensionless temperature variable $\theta(\eta)$ of the form:

$$\theta = \frac{T - T_\infty}{T_w - T_\infty} \tag{4.2}$$

where expression for $T_w - T_\infty$ is given by equation (4.1). Making use of the equation (4.2) in the dimensional energy equation (2.5) we arrive at the following form of non-dimensional thermal boundary layer equation.

$$\theta_{\eta\eta} + \Pr\left(f\theta_\eta - \nu_0 f_\eta \theta\right) = -\Pr E\left\{f_{\eta\eta}^2 + 2Mn^2\left(f_\eta^2 + E_1^2 + 2E_1 f_\eta\right)\right\} \tag{4.3}$$

where $\Pr = \dfrac{\gamma}{\alpha}$ is the Prandtl number and $E = \dfrac{U_0^2}{c_p T_0}\left(\dfrac{U_w}{U_0}\right)^{\frac{4-\nu_0}{2}}$ is the Eckert number.

Temperature boundary conditions of the equation (4.1) take the following non-dimensional form.

$$\theta(0) = 1, \qquad \theta(\infty) = 0 \tag{4.4}$$

The solution of the equation (4.3) subject to the boundary conditions of the equation (4.4) is obtained numerically by applying the method of fifth order Runge-Kutta-Fehlberg method with shooting and outline of the numerical solution procedure is described in the next section.

Case B: prescribed boundary heat flux (PHF)

In order to solve the thermal boundary layer equation (2.5) in PHF case we consider variable heat flux boundary condition of the following form:

$$\begin{aligned} -k\left(\frac{\partial T}{\partial y}\right)_w &= T_1 Exp\left(\frac{\nu_1 + 1}{2l}\right)x \quad at \quad y = 0 \\ T \to T_\infty & \qquad\qquad as \quad y \to \infty \end{aligned} \tag{4.5}$$

where A_1 is the parameter of temperature distribution on the stretching surface and T_∞ is the temperature far away from the stretching sheet. We have considered the above form of quadratic power law heat flux boundary condition on stretching sheet in order to obtain local similar solution of the equation (2.5).

As we look for local similar equation from the thermal boundary layer equation (2.7) we define dimensionless temperature variable $g(\eta)$ of the form:

$$g(\eta)=\frac{T-T_\infty}{\frac{T_1}{k}\sqrt{\frac{2\gamma l}{U_0}}Exp\left(\frac{\nu_1 x}{2l}\right)} \tag{4.6}$$

Making use of the equation (4.6) in the dimensional energy equation (2.5) we arrive at the following form of non-dimensional thermal boundary layer equation in PHF case.

$$g_{\eta\eta}+\Pr\left(fg_\eta-\nu_1 f_\eta g\right)=-\Pr E\left\{f_{\eta\eta}^2+2Mn^2\left(f_\eta^2+E_1^2+2E_1 f_\eta\right)\right\} \tag{4.7}$$

where $\Pr=\dfrac{\gamma}{\alpha}$ is the Prandtl number and $E=\dfrac{U^2{}_0 k}{c_p T_1\sqrt{\dfrac{2\gamma l}{U_0}}}\left(\dfrac{U_w}{U_0}\right)^{\frac{4-\nu_1}{2}}$ is the Eckert number in PHF case.

In view of the transformation equation (4.6) the temperature boundary conditions of the equation (4.5) take the following non-dimensional form.

$$g_\eta(0)=-1, \qquad g(\infty)=0. \tag{4.8}$$

The solution of the equation (4.7) subject to the boundary conditions of the equation (4.8) is obtained numerically by applying the fifth order Runge-Kutta-Fehlberg integration scheme with shooting and outline of the numerical solution procedure is described in the next section.

5. Numerical Solution

The stream function equations (3.7)-(3.8) are highly nonlinear quasi-ordinary differential equations and temperature equations (4.3), (4.7) in PEST and PEHF cases respectively are non-homogeneous quasi-ordinary differential equations with variable coefficients. Solutions of these equations are obtained numerically by employing most efficient numerical shooting technique with fifth order Runge-Kutta-Fehlberg integration scheme. In order to apply this method it is most important to choose the appropriate finite values of $\eta\to\infty$. To select η_∞ for the boundary value problem of Eqs. (3.7) and (3.9) we begin with some initial guess value and solve the problem with some particular set of parameters to obtain $f_{0\eta\eta}(0)$. The solution process is repeated with another large values of η_∞ until two successive values of $f_{0\eta\eta}(0)$ differ only after desired significant digit. The last value of η_∞ is chosen as appropriate finite value of the limit $\eta \to \infty$ for that particular set of parameters to solve the unknown $f_0(\eta)$. Similar procedure is applied to obtain the finite value of η_∞ for the problems of Eqs. (3.8) and (3.10) and Eqs. (4.3)-(4.4) involving unknowns $f_1(\eta)$ and $\theta(\eta)$ respectively. For different set of parameters the appropriate finite values of η_∞ are different.

The coupled boundary value problems of (i) Eqs. (3.7) and (3.9), (ii) Eqs. (3.8) and (3.10) and (iii) Eqs. (4.3) and (4.4) or Eqs. (4.7-4.8) are solved numerically following the method of superposition (Na [1979]). In this method the third order non-linear equations (3.7) and (3.8) and second order equation (4.3) have been reduced to a system of eight simultaneous ordinary differential equations for which five initial conditions (Eqs. (3.9), (3.10) and (4.4) or (4.8)) are prescribed. WE convert this system into a system of initial value problem by employing numerical shooting technique with fifth order Runge-Kutta-Fehlberg integration scheme where three infinity boundary conditions have been utilized to generate three more initial conditions. Once we know all the eight initial conditions we solve this system of simultaneous equations by employing fifth order Runge-Kutta-Fehlberg integration scheme applicable to the system of eight simultaneous first order differential equations. For better approximation of the solutions we employ Newton's iteration.

6. Results and Discussion

Numerical results are computed and they are presented in the Figs. 2-15 for the case of small viscoelastic parameter k_1^* in order to gain information on the effects of all the physical parameters on heat transfer characteristics. Figs 2-3 represent the stream function and velocity profiles, Figs. 4-5 represent skin-friction profiles, Figs 6-9 represent temperature profiles in PST case and Figs 10-13 represent temperature profile in PHF case for various values of heat transfer controlling parameters. The physical layout of the boundary layer in presence of electric field which develops near the slit is depicted in the Fig. 2. Analysis of the figure shows that the effect of increasing values of viscoelastic parameter k_1^* and Hartmann number Mn is to shift the position of streamlines towards the stretching boundary which signifies that there will be a reduction of thickness of the momentum boundary layer. The effect of viscoelastic parameter k_1^* and Hartmann number Mn is seen to decrease the boundary layer velocity throughout the boundary layer but more significantly little away from the stretching sheet. It is to be noted that this reduction of velocity with the increase of viscoelastic parameter k_1^* is more significant in absence of magnetic field.

Fig. 3. represents graph of velocity profile $f_\eta(\eta)$ and stream function $f(\eta)$ for various values of viscoelastic parameter k_1^* and local normal electric parameter E_1 in presence of a uniform transverse magnetic field. Analysis of the figure shows that the effect of local normal electric parameter E_1 is to shift the streamlines towards boundary stretching sheet in both the cases of viscous and viscoelastic fluids flow. The magnitude of shifting of streamlines asymptotically increases with the increase of distance from the stretching sheet. This is in conformity with the realistic situation. The analysis of the figure shows that the effect of local normal electric parameter E_1 on velocity is to decrease its value throughout the boundary layer, more significantly little away from the stretching sheet in the both cases of viscous and viscoelastic fluid flows. However, this reduction velocity with the increase of local normal electric field is more in case of viscous fluid flow compared to viscoelastic fluid flow. This is because Lorentz force which arises due to electric field is higher in magnitude in case of viscous fluid flow than in the case of viscoelastic fluid flow. This force acts as a decelerating force and thereby increases frictional resistance more in case of viscous fluid flow.

Fig. 4 depicts the graphs of skin-friction parameter C_f vs. local viscoelastic parameter k_1^* for different values of local Reynolds number Re_x and Hartmann number Mn in presence of local normal electric field. Analysis of the figure reveals that the effect of increasing the values of Hartmann number Mn is to increase skin-friction coefficient C_f. This increase of skin-friction is the consequence of Lorentz force generated by interaction of flow velocity and magnetic field and it is directed towards up-stream. This change of skin-friction parameter with the change of Hartmann number is only significant in case of fluid flow having smaller viscoelasticity.

Fig. 5 represents the graphs of skin-friction coefficient C_f vs. local viscoelastic parameter k_1^*, for different values of local Reynolds number Re_x and local normal electric parameter E_1 in presence of uniform transverse magnetic field. A careful analysis of the graphs shows that the skin-friction coefficient C_f would decrease with the increase of the values of local Reynolds number Re_x. However the increase of the values of local normal electric parameter E_1 would increase the values of skin-friction parameter C_f. This is due to the fact that the Lorentz force generated by electric field coupled with the reduced magnitude of viscous force decelerates the flow in the down stream direction. It is interesting to observe that skin-friction coefficient C_f decreases linearly with the increase of viscoelastic parameter k_1^*. Separation of boundary layer occurs for the value of viscoelastic parameter $k_1^* = \frac{1}{3}$ which is independent of the values of local Reynolds number Re_x, Hartmann number Mn and local normal electric parameter E_1. This clearly demonstrates that the effect of inertia force and electromagnetic force will be present only for such flow of fluid having smaller viscoelasticity.

The effects of various values of viscoelastic parameter k_1^*, Hartmann number Mn, Prandtl number $\Pr$, local normal electric parameter E_1 and Eckert number E on temperature profile are plotted in the Figs. 6-9 for PST case. Fig. 6 demonstrates the graphs of non-dimensional temperature profile $\theta(\eta)$ for different combination of the values of viscoelastic parameter k_1^*, Hartmann number Mn and Prandtl number $\Pr$. Analysis of the figure shows that, in absence of viscous dissipation and local normal electric field, the effect of Prandtl number $\Pr$ is to decrease temperature in the boundary layer region. Whereas, the effect of increasing values of Hartmann number Mn is to increase temperature in the boundary layer region. The boundary layer fluid would attain maximum temperature if the fluid flow is viscoelastic with low Prandtl number and it is exposed under the influence of a transverse magnetic field.

Fig. 7 depicts the dimensionless temperature profile for the situation when viscous dissipation is accounted but electric field is absent ($E = 0$). In presence of viscous dissipation, it is seen from the figure that there would be higher temperature throughout the boundary layer in case of a viscoelastic fluid compared to viscous fluid. It is also noticed that the effect of Prandtl number is to decrease temperature throughout the boundary layer in both the cases of viscous and viscoelastic fluid in absence of magnetic field. However, in presence of magnetic of magnetic field, the effect of increasing values of Prandtl number is to increase temperature near the stretching sheet and to decrease the same away from the same. The combined effect of increasing values of Prandtl number $\Pr$ and Hartmann number Mn is to increase temperature near the stretching sheet significantly.

Fig. 8 represents the variation of temperature with the change of Hartmann number Mn, Prandtl number $\Pr$ for different values of viscoelastic parameter k_1^* when both viscous dissipation and electric field are taken into consideration. This figure is plotted for the same set of data as that of Fig. 7 except for local normal electric field parameter E_1=0.2. Analysis of this figure reveals the similar feature of temperature profile with the change of Hartmann number, Prandtl number and local viscoelastic parameter. But the region, for which temperature increases, near stretching sheet for increasing Prandtl number in presence of magnetic field, enhances. The combined effect of increasing values of Prandtl number $\Pr$ and Hartmann number Mn is to increase temperature near stretching sheet largely in presence of local normal electric field. This is owing to the reason that local normal electric field acts as a heat source in presence of magnetic field near the boundary sheet whose strength increases with the reduction of values of Prandtl number and decreases with increase of distance from the sheet.

Fig.9 shows the graph of non-dimensional temperature vs. η for different values of local normal electric parameter E_1, Prandtl number $\Pr$ and viscoelastic parameter k_1^* when the effects of viscous dissipation and magnetic field are taken into account. A careful analysis of the figure demonstrates that heat transfer will take place from the adjacent fluid layer to the boundary wall with lower rate when the electric field is not accounted. When the local normal electric field is accounted then the heat transfer process will take place with faster rate. Temperature would attain maximum value in the boundary layer region near stretching sheet. This is because Ohmic dissipation due to electric as well as magnetic field is generated heat in the fluid layer near stretching sheet.

Graphs are plotted for temperature profile in the Figures 10-13, in PHF case. The analysis of the graphs reveal the similar qualitative effects of Hartmann number Mn, Prandtl number $\Pr$, viscoelastic parameter k_1^*, Eckert number E and local normal electric parameter E_1 on temperature profile, but with different magnitudes. However, interestingly we notice that boundary sheet would always attain higher temperature in presence of magnetic and electric fields when viscous dissipation will be taken into account. However, stretching sheet would attain minimum temperature in absence of electromagnetic field when viscous dissipation will not be taken into account. Results of PHF cases shows that there will be always heat transfer from stretching sheet to the adjacent boundary layer region for any values of all the heat controlling physical parameters.

Numerical values of wall temperature gradient $-\theta_\eta(0)$ for different values of Hartmann number Mn, Eckert number E, local normal electric parameter E_1, Prandtl number Pr and viscoelastic parameter k_1^* are computed and those are recorded in the Table 1. Analysis of the tabular data shows that magnetic field and viscous dissipation reverses the direction of heat transfer across the stretching sheet. The effect of increasing values of Prandtl number and local normal electric field is to increase the rate of heat transfer. The effect of viscoelastic parameter k_1^*, in absence of local normal electric parameter E_1, is to reduce the rate of heat transfer. Whereas, the effect of viscoelastic parameter k_1^*, in presence of local electric parameter E_1, is also to increase the rate of heat transfer across the stretching sheet.

7. Conclusion

A numerical local similar solution has been presented on momentum and heat transfer characteristics in an incompressible electrically conducting viscoelastic boundary layer fluid flow over an exponentially stretching sheet where the fluid is influenced by a transverse magnetic field and normal electric field by employing fifth order Runge-Kutta –Fehlberg integration scheme with shooting . Highly non-linear fourth order stream function equation has been solved using three boundary conditions only following the method employed by Rajagopal et al. [1984] and Mahapatra and Gupta [2004] which uses series expansion of stream function $f(\eta)$ up to first order term of viscoelastic parameter k_1^*. The results are analysed for the situation when stretching boundary is prescribed by non-isothermal prescribed surface temperature (PST) and variable boundary heat flux which vary exponentially with the flow directional coordinate x. The effects of various physical parameters like viscoelastic parameter, Prandtl number, local Reynolds number, Eckert number, Hartmann number and local normal electric parameter on various momentum and heat transfers characteristics are analyses. Some of the important conclusions of this analysis are

(i) The effect of increasing values of viscoelastic parameter k_1^* and Hartmann number Mn is to shift the position of streamlines towards the stretching boundary and to decrease the boundary layer velocity throughout the boundary layer but more significantly little away from the stretching sheet.

(ii) The effect of local normal electric parameter E_1 is to shift the streamlines towards boundary stretching boundary. The magnitude of shifting of streamlines asymptotically increases with the increase of distance from the stretching sheet. The effect of local normal electric parameter E_1 on velocity is to decrease its value throughout the boundary layer, more significantly little away from the stretching sheet. This reduction velocity with the increase of local normal electric field is more in case of viscous fluid flow compared to viscoelastic fluid flow.

(iii) The skin-friction coefficient C_f would decrease with the increase of the values of local Reynolds number Re_x and increase with the increase of the values of local normal electric parameter. The skin-friction coefficient C_f decreases linearly with the increase of viscoelastic parameter k_1^*.

(iv) In presence of viscous dissipation, it is seen that there would be higher temperature throughout the boundary layer in case of a viscoelastic fluid compared to viscous fluid. The effect of Prandtl number Pr is to decrease temperature throughout the boundary layer in absence of magnetic field. However, in presence of magnetic of magnetic field, the effect of increasing values of Prandtl number is to increase temperature near the stretching sheet and to decrease the same away from the same.

(v) The combined effect of increasing values of Prandtl number Pr and Hartmann number Mn is to increase temperature near stretching sheet largely in presence of local normal electric field

(vi) When the local normal electric field is accounted then the heat transfer takes place from the adjacent fluid layer to the boundary wall with faster rate.

(vii) In PHF cases, heat transfer from stretching sheet to the adjacent boundary layer region takes place for any values of all the heat controlling physical parameters.

(viii) Magnetic field and viscous dissipation can reverse the direction of heat transfer across the stretching sheet.

(ix) The effect of viscoelastic parameter k_1^*, in presence of local electric parameter E_1, is to increase the rate of heat transfer across the stretching sheet.

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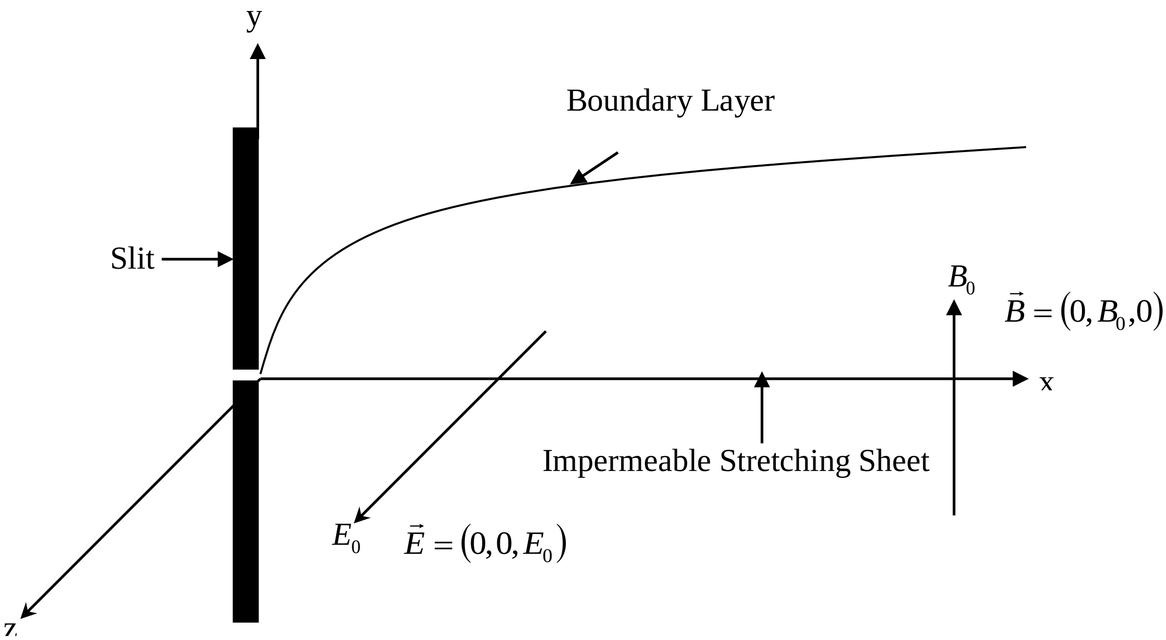

Fig.1. Boundary layer over an impermeable exponentially stretching sheet with electro- magnetic field

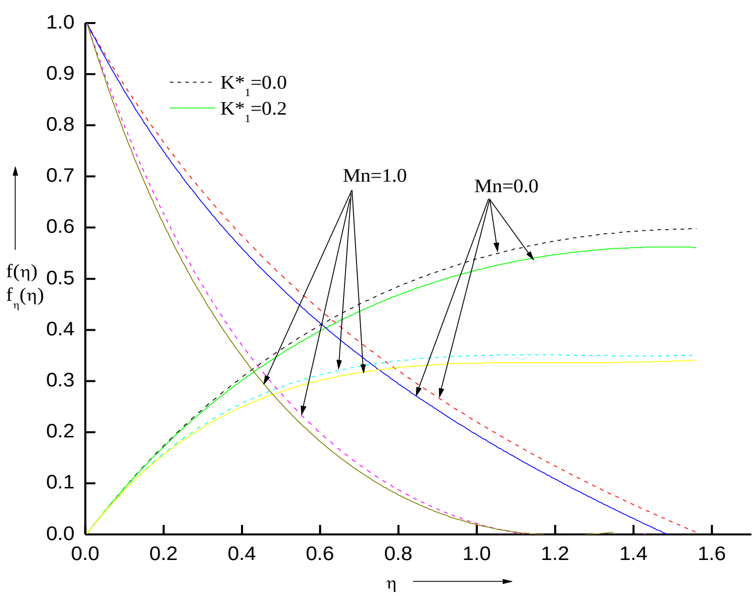

Fig.2. Graph of stream function f(η) and velocity function $f_\eta(\eta)$ vs. η for different values of viscoelastic parameter k^*_1 and Hartmann number Mn when E_1=0.4

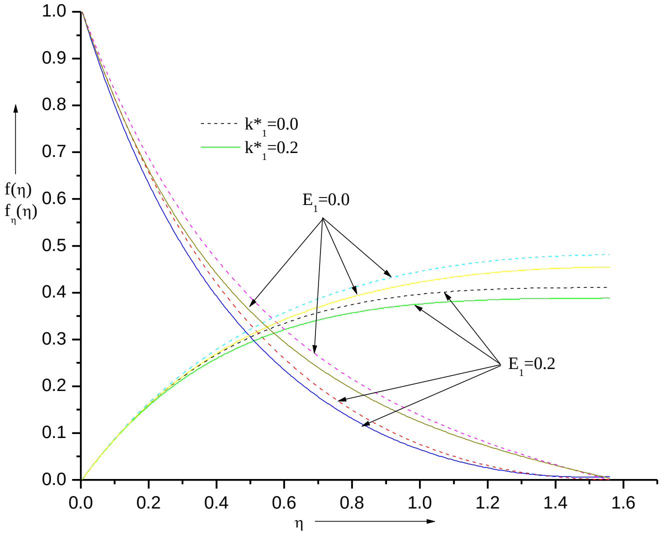

Fig.3. Graph of stream function $f(\eta)$ and velocity function $f_\eta(\eta)$ vs. η for different values of viscoelastic parameter k^*_1 and local electric parameter E_1 when Mn=1.0.

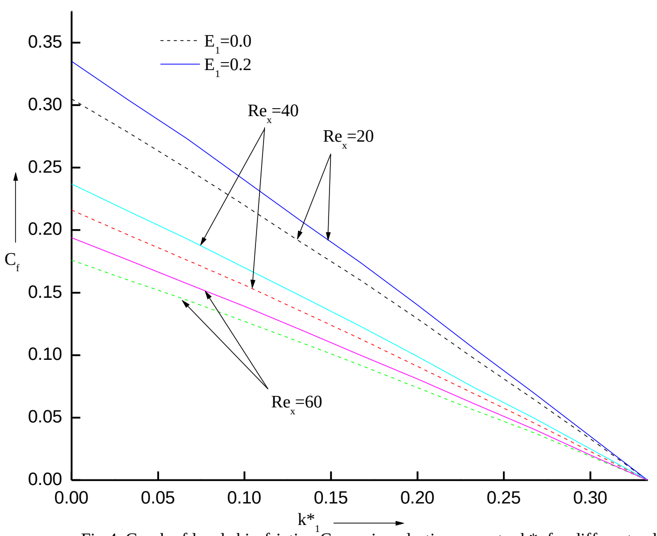

Fig.4. Graph of local skin-friction C_f vs. viscoelastic parameter k^*_1 for different values of local Reynolds number Re_x and local electric parameter E_1 when Mn=1.0.

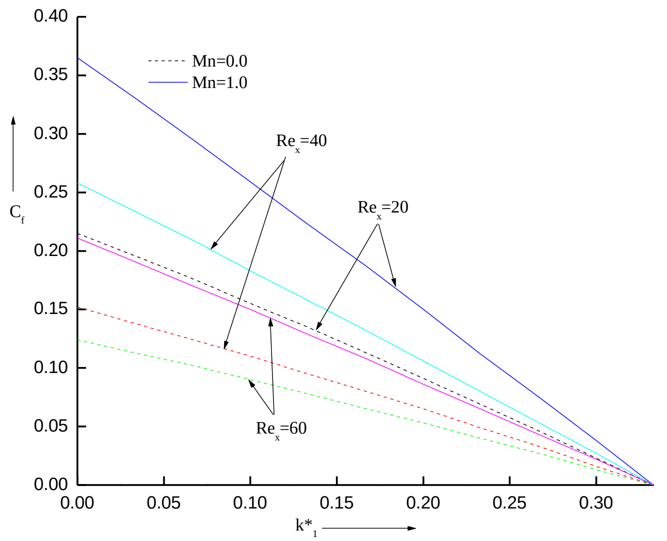

Fig.5 Graph of local skin-friction C_f vs. viscoelastic parameter k^*_1 for different values of local Reynolds number Re_x and Hartmann number Mn when E_1=0.4

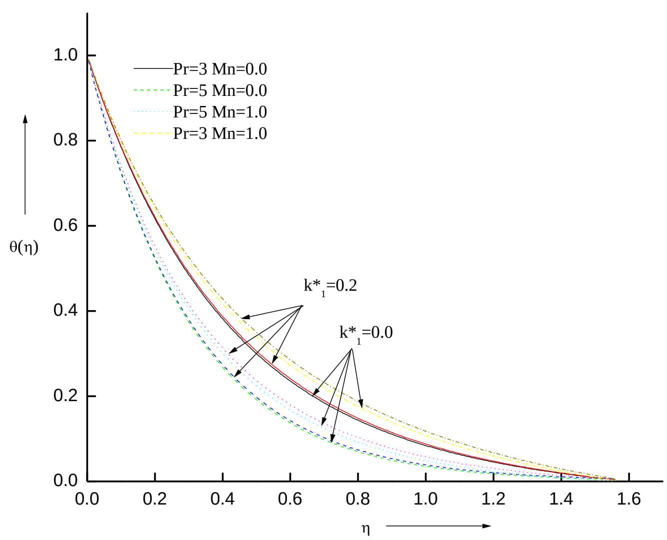

Fig.6 Dimensionless temperaure profile$\theta(\eta)$ for various values of Prandtl number Pr and local Hartmann number Mn when E=0.0 and E_1=0.0 in PST case.

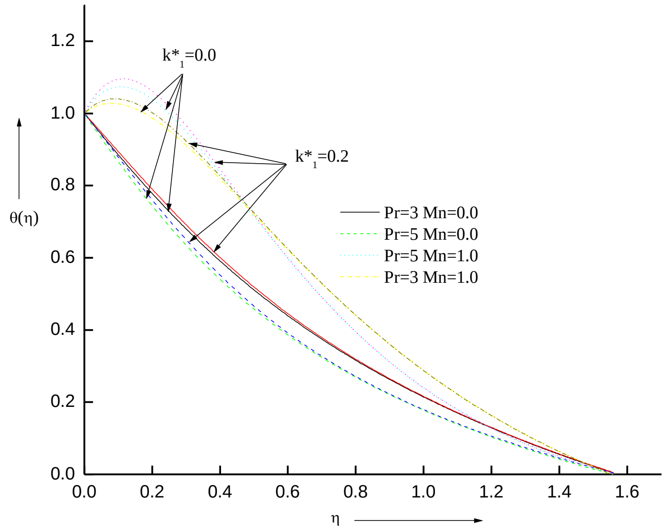

Fig.7 Dimensionless temperature profile θ(η) for various values of Prandtl number Pr and Local Hartmann number Mn when E=1.0 and E_1=0.0 in PST case.

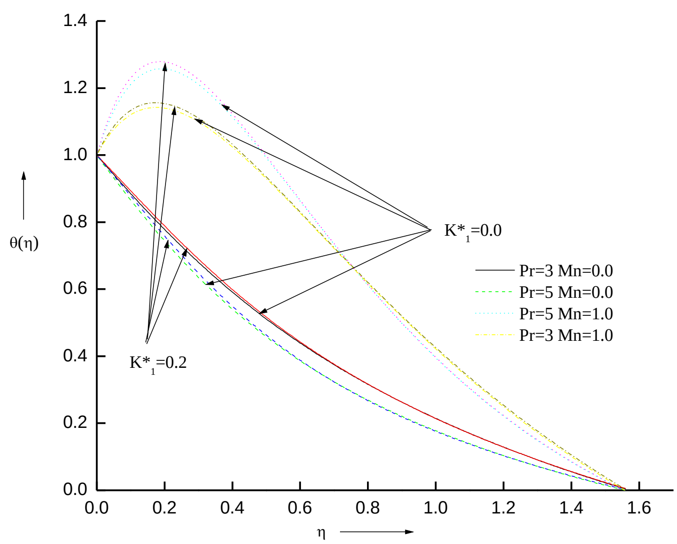

Fig. 8 Dimensionless temperature profile θ(η) for various values of Prandtl number Pr and local Hartmann number Mn when E=1.0 and E_1=0.2 in PST case.

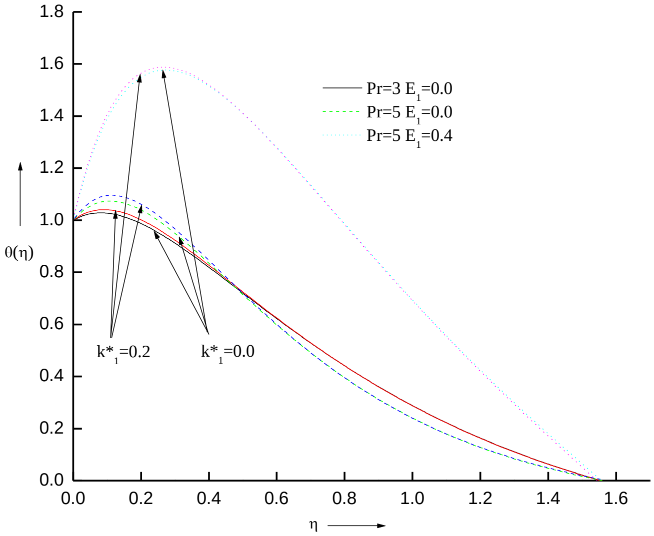

Fig. 9 Dimensionless temperature profile θ(η) for various values of Prandtl number Pr and local Electric parameter E_1 when E=1.0 and Mn=1.0

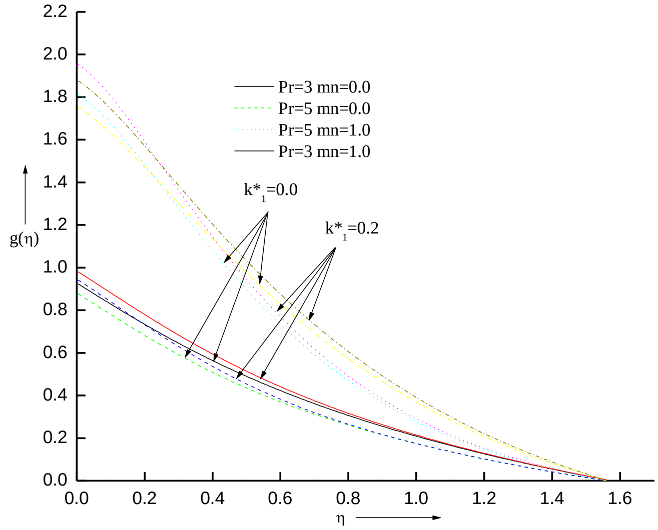

Fig.11 Dimensionless temperature profile g(η) for various values of Prandtl number Pr and local Hartmann number Mn when E=1.0 and E_1=0.0 in PHF case.

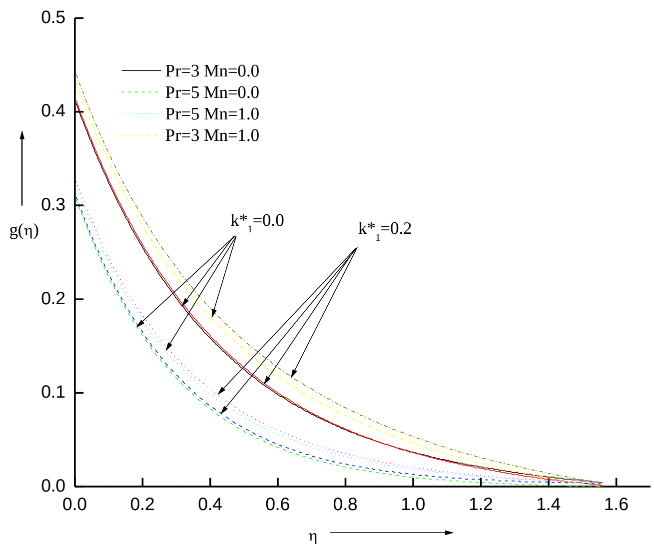

Fig.10 Dimensionless temperature profile g(η) for various values of Prandtl number Pr and local Hartmann number Mn when E=0.0 and E_1=0.0 In PHF case.

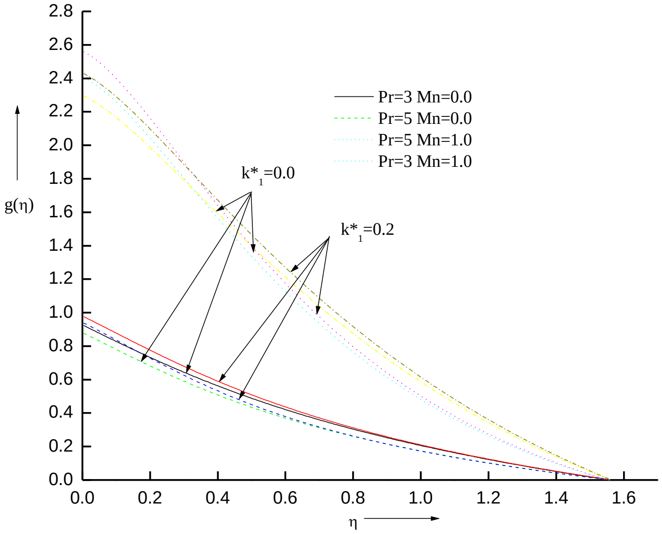

Fig.12 Dimensionless temperature profile g(η) for various values of Prandtl number Pr and local Hartmann number Mn when E=1.0 and E_1=0.2 in PHF case.

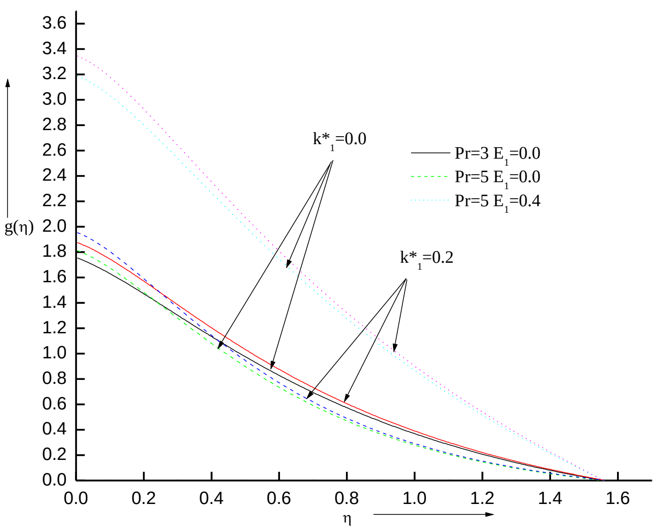

Fig. 13 Dimensionless temperature profile g(η) of various values of Prandtl number Pr and local electic parameter E_1 when E=1.0 and Mn=1.0 in PHF case.