



# Softly Int-normal spaces Department of Mathematics

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**Abstract:** In this paper, we establish and study a new class of softly normal called softly Int-normal spaces. The relationships among normal, ii-normal and Int-normal spaces, softly normal spaces, softly ii-normal spaces are investigated. Moreover, we established some functions related with softly Int-normal spaces and obtain some preservation theorems of softly Int-normal spaces.

**Keywords:** ii-open, softly Int-normal, normal spaces.

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## 1. Introduction

In 1937, Stone [6] established the concept of regular-open sets. In 2019, Mohammed and Abdullah. [1] established the concepts of ii-open, i-open sets and obtained their properties. In 2015, Sharma and Kumar [5] established the notions of softly normal in topological spaces and its properties. In 2020, Hamant Kumar [4] established the concepts of ii-normal spaces and its properties.

## 2. Preliminaries

**2.1 Definition.** A subset  $A$  of a topological space  $X$  is said to be

1. **regular open** [6] if  $A = \text{int}(\text{cl}(A))$ .

2. **INT-open** [1] if

(a)  $G \neq \phi, X$ .

(b)  $\text{INT}(A) = G$

3. **ii-open** [1] set if there exist an open set  $G$  belongs to  $\tau$ , such that

(a)  $G \neq \phi, X$

(b)  $A \subset \text{cl}(A \cap G)$

(c)  $\text{INT}(A) = G$ .

4. The complement of regular open ( resp. **ii-open**) set is said to be **regular closed** (resp. **ii-closed, Int-closed**). The intersection of all ii-closed sets containing A is called the **ii-closure of A** and denoted **ii-cl(A)** (resp. **Int-cl(A)**). The union of all ii-open subsets of X which are contained in A is called the **ii-interior of A** and denoted by **ii-int(A)** (resp. **Int-int(A)**).

The family of all regular open (resp. regular closed, ii-open, ii-closed, int-open, int-closed) sets of a space X is denoted by R-O(X) (resp. R-C(X), ii-O(X), ii-C(X), int-O(X), int-C(X)).

We have the following implications for the properties of subsets :

$$\text{open} \quad \Rightarrow \quad \text{ii-open} \quad \Rightarrow \quad \text{int-open}$$

Where none of the implications is reversible as can be seen from the following examples.

**2.2. Example.** Let  $X = \{p, q, r, s\}$  and  $\tau = \{X, \phi, \{q, s\}, \{p, q, s\}, \{q, r, s\}\}$ . Then

1) closed sets in X are  $X, \phi, \{p\}, \{r\}, \{p, r\}$ .

2) int-open sets in X are  $\phi, X, \{p\}, \{q\}, \{p, q\}, \{p, q, r\}, \{p, q, s\}$ .

**2.3. Example.** Let  $X = \{p, q, r, s\}$  and  $\tau = \{\phi, \{p\}, \{r\}, \{p, q\}, \{p, r\}, \{p, q, r\}, X\}$ . Then

1) closed sets in X are  $X, \phi, \{s\}, \{q, s\}, \{r, s\}, \{p, r, s\}, \{q, r, s\}$ .

2) Int-closed sets in X are  $X, \phi, \{p\}, \{q\}, \{r\}, \{s\}, \{p, q\}, \{p, r\}, \{p, s\}, \{q, r\}, \{q, s\}, \{p, q, s\}, \{q, r, s\}$ .

**2.4. Example.** Let  $X = \{p, q, r\}$  and  $\tau = \{X, \phi, \{p\}, \{q\}, \{p, q\}\}$ . Then

1) closed sets in X are  $X, \phi, \{r\}, \{p, r\}, \{q, r\}$ .

2) ii-open sets in X are  $\phi, X, \{p\}, \{q\}, \{p, q\}, \{p, r\}, \{q, s\}$ .

**2.5. Example.** Let  $X = \{p, q, r\}$  and  $\tau = \{\phi, \{p\}, X\}$ . Then

1) closed sets in X are  $X, \phi, \{q, r\}$ .

2) ii-open sets in X are  $\phi, X, \{p\}, \{p, q\}, \{p, r\}$ .

3) Int-open sets in X are  $X, \phi, \{p\}, \{p, q\}, \{p, r\}$ .

Here  $\{a, b\}$  is ii-open as well as int-open but not open.

### 3. Softly ii-Normal Spaces

**3.1. Definition.** A topological space X is said to be **softly ii-normal** [3] if for every pair of disjoint closed subsets H and K, one of which is  $\pi$ -closed and the other is regular closed, there exist disjoint ii-open sets U, V of X such that  $H \subset U$  and  $K \subset V$ .

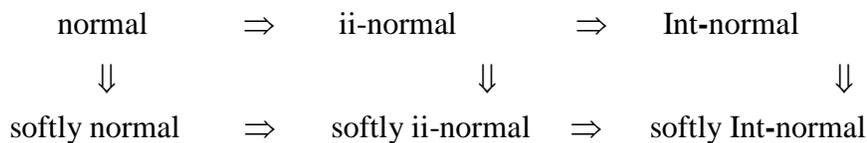
**3.2. Definition.** A topological space X is said to be **int-normal** [2] if for every pair of disjoint closed subsets H, K of X, there exist disjoint INT-open sets U, V of X such that  $H \subset U$  and  $K \subset V$ .

**3.3. Definition.** A topological space X is said to be **ii-normal** [4] if for every pair of disjoint closed subsets H, K of X, there exist disjoint ii-open sets U, V of X such that  $H \subset U$  and  $K \subset V$ .

**3.4. Definition.** A topological space X is said to be **softly Int-normal** (resp. **softly normal** [5]) if for every pair of disjoint closed subsets H and K, one of which is  $\pi$ -closed and the other is regular closed, there exist disjoint int-open sets U, V of X such that  $H \subset U$  and  $K \subset V$ .

### 3.5. Remark.

By the definitions and examples stated above, we have the following diagram:



**3.6. Example.** Let  $X = \{p, q, r, s\}$  and  $\tau = \{\phi, \{p\}, \{q\}, \{r\}, \{p, q\}, \{p, r\}, \{q, r\}, \{p, q, r\}, \{p, q, s\}, X\}$ . The pair of disjoint closed subsets of  $X$  are  $A = \{r\}$  and  $B = \{s\}$ . Also  $U = \{q, r\}$  and  $V = \{p, s\}$  are disjoint int-open sets such that  $A \subset U$  and  $B \subset V$ . Hence  $X$  is softly Int-normal but not normal.

**3.7. Example.** Let  $X = \{p, q, r, s\}$  and  $\tau = \{\phi, \{r\}, \{s\}, \{q, r\}, \{r, s\}, \{p, r, s\}, \{q, r, s\}, X\}$ . The pair of disjoint closed subsets of  $X$  are  $A = \{p\}$  and  $B = \{q\}$ . Also  $U = \{p, r\}$  and  $V = \{q, s\}$  are disjoint int-open sets such that  $A \subset U$  and  $B \subset V$ . Hence  $X$  is softly Int-normal but neither softly normal nor normal.

**3.8. Example.** Let  $X = \{p, q, r, s\}$  and  $\tau = \{\phi, \{q\}, \{s\}, \{q, s\}, \{p, q, s\}, \{q, r, s\}, X\}$ . The pair of disjoint closed subsets of  $X$  are  $A = \{p\}$  and  $B = \{r\}$ . Also  $U = \{p, s\}$  and  $V = \{q, r\}$  are disjoint int-open sets such that  $A \subset U$  and  $B \subset V$ . Hence  $X$  is int-normal but not normal. Since  $U$  and  $V$  are not open.

**3.9. Example.** Let  $X = \{p, q, r, s\}$  and  $\tau = \{\phi, \{p\}, \{q, r, s\}, X\}$ . The pair of disjoint closed subsets of  $X$  are  $A = \{p\}$  and  $B = \{q, r, s\}$ . Also  $U = \{p\}$  and  $V = \{q, r, s\}$  are disjoint ii-open sets such that  $A \subset U$  and  $B \subset V$ . Hence  $X$  is softly normal as well as softly ii-normal. Since  $U$  and  $V$  both are open sets.

**3.10. Example.** Let  $X = \{p, q, r, s\}$  and  $\tau = \{\phi, \{p\}, \{q\}, \{p, q\}, \{p, q, r\}, X\}$ . The Let  $X = \{p, q, r, s\}$  and  $\tau = \{\phi, \{p\}, \{q, r, s\}, X\}$ . The pair of disjoint closed subsets of  $X$  are  $A = \{\phi\}$  and  $B = \{q, r, s\}$ . Also  $U = \{p\}$  and  $V = \{q, r, s\}$  are disjoint int-open sets such that  $A \subset U$  and  $B \subset V$ . Hence  $X$  is softly Int-normal as well as softly ii-normal. Since  $U$  and  $V$  both are open sets.

**3.11. Example.** Let  $X = \{p, q, r, s\}$  and  $\tau = \{\phi, \{p\}, \{q\}, \{p, q\}, \{p, r\}, \{p, q, r\}, X\}$ . The pair of disjoint closed subsets of  $X$  are  $A = \{\phi\}$  and  $B = \{s\}$ . Also  $U = \{p\}$  and  $V = \{q, r, s\}$  are disjoint int-open sets such that  $A \subset U$  and  $B \subset V$ . Hence  $X$  is softly Int-normal but not softly ii-normal.

**3.12. Theorem.** For a topological space  $X$ , the following are equivalent:

- (1)  $X$  is softly Int-normal.
- (2) For each  $\pi$ -closed set  $G$  and each regularly open set  $H$  with  $G \subset H$ ,
- (3) There exist an int-open set  $U$  s. t.  $G \subset U \subset \text{ii-cl}(U) \subset H$ . For each regularly closed set  $G$  and each  $\pi$ -open set  $H$  with  $G \subset H$ , there exist an int-open set  $U$  s. t.  $G \subset U \subset \text{int-cl}(U) \subset H$ .
- (4) For any pair possessing of disjoint closed set  $G$  and  $H$ , one of which is  $\pi$ -closed and the other is regularly closed, there exist int-open sets  $U$  and  $V$  s. t.  $G \subset U$  and  $H \subset V$  and also  $\text{int-cl}(U) \cap \text{Int-cl}(V) = \phi$ .

Proof: (1)  $\Rightarrow$  (2).

Suppose (1). Let  $G$  and  $H$  are any pair of disjoint closed sets one of which is  $\pi$ -closed and the other is regularly open s. t.  $G \subset H$ . Then  $G \cap (X - H) = \phi$  where  $(X - H)$  is regularly closed. Then there exist disjoint int-open

sets  $U, V$  s. t.  $G \subset U$  and  $(X - H) \subset V$ . Because  $U \cap V = \phi$ , then  $\text{Int-cl}(U) \cap V = \phi$ . So  $\text{Int-cl}(U) \subset (X - V) \subset (X - (X - H)) = H$ . Hence  $G \subset U \subset \text{Int-cl}(U) \subset H$ .

(2)  $\Rightarrow$  (3).

Suppose (2).

Consider  $G$  is any regularly closed set and  $H$  is any  $\pi$ -open set s. t.  $G \subset H$ . Then  $(X - H) \subset (X - G)$ , where  $(X - H)$  is  $\pi$ -closed and  $(X - G)$  is regularly open. Therefore by (2), there exist an int-open set  $M$  s. t.  $(X - H) \subset M \subset \text{ii-cl}(M) \subset (X - G)$ . So  $G \subset (X - \text{Int-cl}(M)) \subset (X - M) \subset H$ . Therefore, we suppose  $U = (X - \text{Int-cl}(M))$ , which is int-open and since  $M \subset \text{Int-cl}(M)$ , then  $(X - \text{Int-cl}(M)) \subset (X - M)$ . Therefore  $U \subset (X - M)$ , thus  $\text{Int-cl}(U) \subset \text{Int-cl}(X - M) = (X - M) \subset H$ .

(3)  $\Rightarrow$  (4).

Suppose (3).

Consider  $G$  and  $H$  are any regular closed and  $\pi$ -closed set with  $G \cap H = \phi$ . Then  $G \subset (X - H)$  where  $(X - H)$  is  $\pi$ -open. By (3), there exists an int-open set  $U$  s. t.  $A \subset U \subset \text{Int-cl}(U) \subset (X - H)$ . But  $\text{Int-cl}(U)$  is int-closed, using (3), again we have an int-open set  $M$  such that  $G \subset U \subset \text{Int-cl}(U) \subset M \subset \text{Int-cl}(M) \subset (X - H)$ . Suppose  $(X - \text{Int-cl}(M))$ , then  $V$  is int-open set and  $H \subset V$ . We get  $(X - \text{Int-cl}(M)) \subset (X - M)$ , so  $V \subset (X - M)$ , therefore  $\text{Int-cl}(V) \subset \text{Int-cl}(X - M) = (X - M)$ . Then we get  $\text{Int-cl}(U) \subset M$  and  $\text{Int-cl}(V) \subset (X - M)$ . So  $\text{Int-cl}(U) \cap \text{Int-cl}(V) = \phi$ .

(4)  $\Rightarrow$  (1) is obvious.

**3.13. Theorem.** For a topological space  $X$ , the following are equivalent :

(1)  $X$  is softly Int-normal.

(2) For any pair of open sets  $A$  and  $B$ , one of which is  $\pi$ -open and the other is regular open their union is  $X$ , there exist an int-closed sets  $U, V$  such that  $A \subset U$  and  $B \subset V$  and  $U \cup V = X$ .

(3) For each  $\pi$ -closed set  $G$  and each regular open set  $H$  containing  $G$ , there is an int-open set  $V$  such that  $G \subset V \subset \text{int-cl}(V) \subset H$ .

**Proof.** (1)  $\Rightarrow$  (2), (2)  $\Rightarrow$  (3), (3)  $\Rightarrow$  (1).

(1)  $\Rightarrow$  (2). Suppose  $A, B$  are  $\pi$ -open and regular open sets respectively in a softly Int-normal space  $X$  such that  $A \cup B = X$ . Then  $X - A$  and  $X - B$  are  $\pi$ -closed and regular closed sets of  $X$  with  $(X - A) \cap (X - B) = \phi$ . By softly Int-normal of  $X$ , there exist disjoint int-open sets  $A_1, B_1$  such that  $X - A \subset A_1$  and  $X - B \subset B_1$ . Suppose  $U = X - A_1$  and  $V = X - B_1$ . Then  $U$  and  $V$  are int-closed sets such that  $U \subset A, V \subset B$  and  $U \cup V = X$ .

(2)  $\Rightarrow$  (3) and (3)  $\Rightarrow$  (1) are clear. By using previous theorem (3.8), this is easy to produce the following theorem, which is a Urysohns lemma version to softly Int-normal. A proof can be introduced by a similar manner of the normal case.

**3.14. Theorem.** A topological space  $X$  is soft Int-normality iff for each pair of disjoint closed sets  $G$  and  $H$  of which is  $\pi$ -closed and the other is regularly closed, there exist a continuous function  $g$  on  $X$  into  $[a, b]$ , with its usual topology, s. t.  $g(G) = \{a\}$  and  $g(H) = \{b\}$ . As above this can be see that the inverse image of a  $\pi$ -closed set under an open continuous function is  $\pi$ -closed and the inverse image of a regularly closed set under an open continuous function is regularly closed.

**3.15. Theorem.** Suppose  $X$  is a soft Int-normality and  $g : X \rightarrow Y$  is an open continuous one-one function. Then  $g(X)$  is a soft Int-normality.

**Proof:** Suppose  $G$  be any  $\pi$ -closed subset in  $g(X)$  and consider  $H$  be any regularly closed subset in  $g(X)$  s. t.  $G \cap H = \phi$ . Then  $g^{-1}(G)$  is a  $\pi$ -closed set in  $X$ , which is disjoint from the regularly closed set  $g^{-1}(H)$ . Because  $X$  is softly Int-normal, there are two disjoint open sets  $S$  and  $T$  s. t.  $g^{-1}(G) \subset S$  and  $g^{-1}(H) \subset T$ . Because  $g$  is injective and open. Hence the result follows. **Corollary:** Softly Int-normal space is a topological property.

**3.16. Lemma:** Suppose  $N$  be a closed domain subspace of a topological space  $X$ . If  $G$  be an int-open set in  $X$ , then  $G \cap N$  is int-open set in  $N$ .

**3.17. Theorem.** A closed domain subspace of a soft Int-normality is soft Int-normality.

**Proof:** Suppose  $N$  be a closed domain subspace of a soft Int-normality  $X$ . Suppose  $G, H$  be any disjoint closed sets in  $N$  s. t.  $G$  and  $H$  are regularly closed and  $\pi$ -closed respectively. Then  $G$  and  $H$  are disjoint closed sets in  $X$  such that  $G$  is regularly closed and  $H$  is  $\pi$ -closed in  $X$ . By softly Int-normal space of  $X$ , there exist disjoint ii-open sets  $S$  and  $T$  of  $X$  such that  $G \subset S$  and  $H \subset T$ . By using the **Lemma (3.16)**, we get  $S \cap N, T \cap N$  are disjoint int-open sets in  $N$  such that  $G \subset S \cap N$  and  $H \subset T \cap N$ . Therefore,  $N$  is softly Int-normal subspace. Because each closed and open (clopen) subset is a closed domain, then we get the following corollary.

**Corollary:** Softly Int-normal space is a hereditary under the clopen subspaces.

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