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A Review on singularly perturbed problems

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Abstract: In this paper, we review various types of singularly perturbed problems and discuss the methods to solve singularly perturbed problems. It contains an impressive amount of material and is a great introduction to some of the ideas and methods of singular perturbation theory. The results of several recent methods are summarized and recommendations are given for singular perturbed problems. In this review article on singular perturbations in "classical" analysis, we address several important aspects of the theory and provide a few techniques. However, it is hard to offer a comprehensive assessment here due to the variety of issues and the huge volume of literature on the subject. A significant amount of work has been done in the field of single perturbations starting with Prandtl's work. This study restricts its coverage to a few common single perturbation models taken into account by different academics, as well as the methodologies created by multiple researchers between 1980 and 2000.

KEYWORDS: Domain, techniques, Perturbation, parameter, Asymptotic.

INTRODUCTION

The development of science and technology has led to a growing number of practical problems, which have become increasingly complex and require asymptotic methods to solve, for example, the mathematical boundary layer theory or approximation of solutions to various differential equations with large or small parameters. Asymptotic analysis and numerical analysis are the two main methods for resolving singular perturbation problems. There hasn't been much interaction between these approaches because the aim and problem classes are so dissimilar. Asymptotic analysis seeks to understand the qualitative behavior of a family of problems, whereas numerical analysis seeks to provide quantitative information on a specific problem.

The theory of asymptotic analysis for differential operators has been developed mostly for regular perturbations, in which the perturbations are subordinate to the unperturbed operator. The dependent variable undergoes rapid changes in some problems when perturbations are restricted to a very narrow region. Because a small parameter multiplies the highest derivative, these narrow regions are frequently adjacent to the boundary of the domain of interest. As a result, they are called boundary layer.

Friedrichs and Wasow [2] originally used the phrase "singular perturbations" in a presentation that delivered at a seminar on non-linear vibrations at New York University. Layers are frequently present in the solution of singular perturbation issues. When the term "boundary layer" was first coined by Prandtl [1], Wasow's work led to its more widespread use. Pearson [4] was one of the first to suggest using the finite-difference method to solve problems involving singular perturbation. Cubic splines and finite element techniques have recently been explored by researchers to solve singular perturbation problems.

RESEARCH METHODOLOGY

The present study was based on Secondary data.

- **Secondary data:** The secondary data will be collected from published books, journals, research papers, internet etc.

NUMERICAL TECHNIQUES FOR SOLVING SINGULARLY PERTURBED PROBLEMS

Problem 1[6]

Consider the singularly perturbed self-adjoint boundary value problem

$$Lu(x) = -\varepsilon^2 u''(x) + b(x)\, u(x) = f(x), \qquad x \in D = (0, 1) \qquad (1)$$

$$u(0) = A \;, \quad u(1) = B \qquad (2)$$

Where $\varepsilon > 0$ is a small parameter, and b and f are sufficiently smooth functions such that $b(x) \geq \beta > 0$ on $D = [0, 1]$. Under these assumptions, the above problem has a unique solution $u(x) \in c^2(D)$. In general, the solution u(x) may exhibit two boundary layers of exponential type at both end-points x = 0, 1.

The major goal of this study is to solve singular perturbation problems of the forms (1) and (2) after adequate modification using higher-order spline techniques of regular problems. The domain is split into three sub domains—two boundary-layer, sub domains and one regular sub domain—and the boundary layer problem is appropriately transformed using stretching variables to become a regular problem. The numerical solution is then obtained in three sub domains using a difference approach. An asymptotic approximation solution yields the boundary conditions at the transition sites. As a result, a higher-order accuracy of $O(h^4)$ was established, and since boundary layers and common sub domain problems were involved, parallel computing techniques could be used to speed up calculation.

Problem 2 [7]

Consider a system of linear singular perturbed boundary value problems (SPBVPs) for second order impulsive ordinary differential equations (IODEs) with a small parameter multiplying the highest derivative as follows:

$$\varepsilon\, y''(x) + b(x)y'(x) + c(x)y(x) = f(x)$$

$$x \in J, \;\; x \neq 0, 1, x_k \qquad (k = 1, 2, \ldots, m),$$

$$\Delta y|x = x_k = I_k(y(x_k)) \qquad (k = 1, 2, \ldots, m)$$

$$\Delta y\, |x = x_k = N_k(y'(x_k)) \qquad (k = 1, 2, \ldots, m),$$

$$y(0) = \alpha, \quad y(1) = \beta$$

Where ε is a small positive parameter ($0 < \varepsilon << 1$), α and β are known constants,

$$I_k, N_k \in (R, R), \quad J = [0, 1], \qquad 0 < x_1 < x_2 < \cdots < x_m < 1.$$

b(x), c(x), and f (x) are sufficiently continuously differential functions on J , and $\Delta y(x_k)$ and $\Delta y'(x_k)$ represent increments of y(x) and $y'(x_k)$, respectively, at $x = x_k$.

In [7], the authors propose a general method in which the defined interval is divided into a series of subintervals and the solution was obtained on each subinterval. On the subinterval of the boundary layer, an expansion is obtained by using matching techniques. Then, using impulsive and boundary conditions, solutions are obtained on other subintervals. Finally, several linear and non-linear singularly perturbed impulsive problems are solved using this new approach.

Problem 3 [8]

Consider a family of singularly perturbed two-point singular boundary value problems

$$-\varepsilon y'' + p(x)\, y' + q(x)\, y = f(x), \qquad 0 < x < 1$$

Subject to the natural boundary conditions $y(0) = A$, $y(1) = B$, where $0 < \varepsilon << 1$, $p(x)$, $q(x)$, and f (x) are bounded continuous functions in (0,1), and A and B are finite constants.

In [8], On a non-uniform mesh, tension spline numerical methods are used to effectively numerically integrate problems involving singularly perturbed two-point singular boundary values. Both problems with rectangular coordinates and those with polar coordinates can be solved using the suggested methods. Numerical evaluations and error analysis are presented to demonstrate the effectiveness of the suggested approach.

Problem 4 [9]

Consider a numerical solution of the singularly perturbed reaction-diffusion problem using grid equidistribution:

$$Lu = -\varepsilon u''(x) + b(x)u(x) = f(x), \qquad x \in (0, 1)$$
$$u(0) = 0,\ u(1) = 0,$$

where $b(x) \geq \beta > 0$ and $b(x)$ and $f(x)$ are assumed to be sufficiently smooth. In particular, the authors are interested in the singularly perturbed regime where $0 < \varepsilon << \beta$.

It is well known that the possibility of steep exponential boundary layers makes it challenging to address this problem using conventional numerical methods. In singularly perturbed reaction-diffusion situations, these layers are also present.

In [9], An equidistribution of a positive monitor function, which is a linear combination of a constant floor and a power of the second derivative of the solution, is used as the foundation for a grid.

The analysis demonstrates how to pick the monitor function so that the precision of the numerical approximation is independent of the magnitude of the singular perturbation parameter.

Many realistic grid adaptation systems employ the equidistribution principles, and the study described in [43] sheds light on the convergence behavior on such grids. We give numerical findings that support the uniform convergence rates.

Problem 5 [10]

Consider a singularly perturbed two-point boundary value problem with a turning point:

$$Ly \equiv \varepsilon y'' + a(x)y' - b(x)y = f(x) \ \ on \quad [p1, p2],$$

$$y(p_1) = \eta_1,\ y(p_2) = \eta_2.$$

This problem is resolved using a non-uniform cubic spline approach. Second-order precision is provided by this method and numerical examples in favor of it are provided.

ADDITIONAL DEVELOPMENTS

Singular perturbation problems are solved using a discretization technique: When the issue is parameter-dependent, this method is helpful. In systems of equation including several unknowns, whether linear or non-linear, it can also be helpful. These systems are frequently solved using iterative techniques. It should be emphasized that the iterative approach has greater power than the discretization approach. Analyzing the dependence of those constants that appear in consistency, stability, and error estimates on this parameter requires caution.

Numerous isolated perturbation problems have exponential solutions in some research articles. So instead of using grid generation, an exponentially fitted approach is utilized, which yields better results. Theoretical error estimation for both linear and non-linear issues is done using fitted mesh methods. However, they cannot be expanded to multi-dimensions and their implementation is computationally expensive, particularly for non-linear situations. Using conventional discretization on a carefully selected non uniform grid is an alternate method. Any non-uniform grid approach, though, needs a lot of a priori knowledge about the existence, position, height, and width of the layers in order to be used successfully. Therefore, it is vital to create adaptable algorithms that can avoid the requirement for unrealizable amounts of a priori knowledge about the solution.

STANDARD SINGULAR PERTURBATION MODEL

M.K. Kadalbajoo and K.C. Patidar / Appl. Math. Comput. 134 (2003) 371-429 373, discuss how the development of small parameter methods allowed the boundary-layer theory to be used effectively in a variety of applied mathematics fields, including fluid mechanics, fluid dynamics, elasticity, quantum mechanics, plasticity, chemical reactor theory, aerodynamics, plasma dynamics, magneto hydrodynamics, rarefied-gas dynamics.

LITERATURE REVIEW

In 2011, Kumar [11] studied methods for solving singular perturbation problems arising in science and engineering. Author has been pointed out tha the ever-increasing number of industrial applications of singular perturbation problems and the richness of behavior of the governing equations make this area a particularly rewarding one for mathematicians, engineers, and industrialists alike.

In 2012, Teixeira & da Silva [12], analyzed regularization and singular erturbation techniques for non-smooth systems. Authors presented a compact survey of the geometric/qualitative theoretical features of non-smooth systems and pointed out that the survey was illustrating that this field is still in its early stages but is enjoying growing interest.

In 2018,] Gadisa et al.[13],studied fourth order numerical method for singularly erturbed delay differential equations. Authors presented fourth order numerical ethod for solving singularly perturbed delay reaction-diffusion equations with twin layers and oscillatory behavior. To demonstrate the efficiency of the method, four model examples without exact solutions have been considered for different values of the perturbation parameter $\in$ and delay parameter δ.Authors investigated the stability and $\in$ uniform convergence of the method.

In 2020, Yamac et al.,[14] presented a numerical scheme for semi linear singularly perturbed reaction-diffusion problems. Authors presented a Numerov's scheme to solve a class of singularly perturbed semi linear reaction-diffusion problem and introduced computational technique based Numerov's cheme on a uniform mesh. Uniform convergence of the method is demonstrated with respect to the parameter of perturbation.

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