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Roots of transcendental equations

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Abstract: This article will find and prove a formula for solutions of the equations of the form $ax^{\alpha} + bx^{\beta} + c = 0$. Where a, b, c, α, β are real numbers and $\alpha > \beta$.

1. Introduction

The Egyptian Mathematician Berlin Papyrus, dating back to the middle kingdom (2050 BC to 1650 BC) and Greek Mathematician Euclid (circa 300 BC) used geometric methods to solve quadratic equations. In his work Arithmatica, the Greek Mathematician Diophantus (circa 250 AD) solved quadratic equations with a method more recognizably algebraic than the geometric algebra of Euclid, his solutions gives only one root, even when both roots exists. Indian Mathematician Brahmgupta described the quadratic formula in his treatise Brãhmãsphutãsiddhãnta published in 628 AD, although he described it in words, according to this solution of the quadratic equation $ax^2 + bx + c = 0$ is given by

$$x = \frac{(\sqrt{(4ac + b^2)} - b)}{2a}.$$

Another Indian Mathematician Sridhrãcãryya (870-930 AD) came up with similar formula having no consideration for both the roots. The quadratic formula covering all cases was first obtained by Simon Stevin in 1594 AD. In 1637 Rene Descartes published La Geometrie containing special cases of quadratic formula in the form we know today. Abel's impossibility theorem states that there is no solution in radicals to general polynomial of degree Five of more. Galois theory for non solvable quintic shows that

$$x^5 - x - 1 = 0.$$

is the simplest equation that can not be solved in radicals. There is no theoretical development to find roots of the equations of the form

$$a_n x^{\alpha_n} + a_{n-1} x^{\alpha_{n-1}} + \cdots a_1 x^{\alpha_1} += 0,$$

Where a_0 , $a_1, \dots, a_n$; $\alpha_1, \dots, \alpha_n$ are real numbers and $\alpha_n > \alpha_{n-1}, \dots > \alpha_1$. Equations of the above form are common in theoretical physics. Numerical solutions of such equations, gives no insight of the further evolution of the dynamical systems. This article will find and prove a formula for solutions of the equations of the form

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This paper is in final form and no version of it will be submitted for publication elsewhere.

$$ax^{\alpha} + bx^{\beta} + c = 0 \quad . \tag{1.1}$$

Where a, b, c, α, β are real numbers and $\alpha > \beta$.

2. Main results

Proposition 1. Suppose in equation *(1.1)* α, β are rational numbers, then equation *(1.1)* is a polynomial equation.

Proof: Put $\alpha = p_1/q_1$ and $\beta =$ where p_1, q_1, p_2, q_2 are natural numbers, also put

$x^{\frac{1}{q_1 q_2}} = y$, get

$$x^\alpha = x^{p_1/q_1} = y^{q_2 p_1};\ x^\beta = x^{p_2/q_2} = y^{q_1 p_2} \tag{2.1}$$

from (1.1) and (2.1) get

$$a y^{q_2 p_1} + b y^{q_1 p_2} + c = 0. \tag{2.2}$$

equation (2.2) is a polynomial equation. Since $\alpha > \beta$, therefore $q_2 p_1 > q_1 p_2$, hence equation (2.2) has degree $q_2 p_1$.

Remark 1. *If equation (2.2) has degree* $q^2 p^1 > 5$, *then it could not be solved in radical as follows from the introduction part.*

Remark 2. *If* $\alpha > \beta$ *in (1.1) are not rational then equation (1.1) turns to be a transcendental equation as it could not be reduced to a polynomial equation by fundamental algebraic operations of addition, subtraction, division and multiplication.*

Theorem 1. *For* $0 < \alpha < 1$, *if* z_0 *is root of equation* $y^\alpha + y + c = 0$, *then* $b^{\frac{-1(\ln b - \ln a)}{(1-\alpha)\ln b}} z_0$ *is root of* $ax^\alpha + bx + D = 0$, *where* $D = c.b^{1-\frac{(\ln b - \ln a)}{(1-\alpha)\ln b}}$.

Proof: Rewrite equation $ax^\alpha + bx + D = 0$ as $ax^\alpha + b^{1-k}.b^k x + D = 0$,where k is a non zero real number. Put $b^k x = y$, then $x = b^{-k} y$, this gives $ax^\alpha = ab^{-\alpha k} y^\alpha$. Thus equation $ax^\alpha + bx + D = 0$ becomes

$$ab^{-\alpha k} y^\alpha + b^{1-k} y + D = 0. \tag{2.3}$$

Choose k so that $ab^{-\alpha k} = b^{1-k}$, this gives

$$k = \frac{(\ln b - \ln a)}{(1-\alpha)\ln b}.$$

Divide equation (2.3) by b^{1-k} , to get

$$y^\alpha + y + c = 0 \tag{2.4}$$

Now if z_0 is root of equation (2.4) then $x = b^{-k} y$, gives that $b^{\frac{-1(\ln b - \ln a)}{(1-\alpha)\ln b}} z_0$ is root of $ax^\alpha + bx + D = 0$.

Theorem 2. For $\left(\frac{\alpha}{\beta}\right) > 1$, if z_0 is root of equation $y^{\frac{\alpha}{\beta}} + y + c = 0$, then $b^{\frac{-1(\ln b - \ln a)}{(\alpha-\beta)\ln b}} z_0$ is root of $ax^\alpha + bx^\beta + D = 0$, where $D = c.b^{1-\frac{\beta(\ln b - \ln a)}{(\alpha-\beta)\ln b}}$.

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Proof: Rewrite equation $ax^\alpha + bx^\beta + D = 0$ as $ax^\alpha + b^{1-k}.b^k x^\beta + D = 0$, where k is a non zero real number. Put $b^k x^\beta = y$, then $x = \left(\frac{y}{b^k}\right)^{\frac{1}{\beta}}$ this gives $ax^\alpha = a.b^{\frac{-k\alpha}{\beta}} y^{\frac{\alpha}{\beta}}$. Thus equation $ax^\alpha + bx^\beta + D = 0$ becomes

$$a.b^{\frac{-k\alpha}{\beta}} y^{\frac{\alpha}{\beta}} + b^{1-k} y + D = 0. \tag{2.5}$$

Choose k so that $a.b^{\frac{-k\alpha}{\beta}} = b^{1-k}$, this gives

$$k = \frac{\beta(\ln b - \ln a)}{(\alpha-\beta)\ln b}.$$

Divide equation (2.5) by b^{1-k}, to get

$$y^{\frac{\alpha}{\beta}} + y + c = 0. \tag{2.6}$$

Now if z_0 is root of equation (2.6) then $x = b^{\frac{-k}{\beta}} y$, gives that $b^{\frac{-1(\ln b - \ln a)}{(\alpha-\beta)\ln b}} z_0$ is root of $ax^{\alpha} + bx^{\beta} + D = 0$. This completes the proof.

Theorem 3. For $0 < \alpha < 1$, equation $ax^{\alpha} + bx - 2c = 0$, has a root at,
$c + \frac{c^{2-\alpha}}{2^{\alpha}\frac{\sin\pi\alpha}{\pi} + c^{1-\alpha}} - c^{\alpha}$, and for $\alpha > 1$, it has a root at $c - \frac{c^{2-\alpha}}{2^{\alpha}\frac{\sin\pi\alpha}{\pi} + c^{1-\alpha}} + c^{\alpha}$,

Proof: Plot $y = x^{\alpha}$ and $y = x^{\frac{1}{\alpha}}, 0 < \alpha < 1, y = x$ and on Cartesian plane (for illustration take $\alpha = \sqrt{(2)}$)

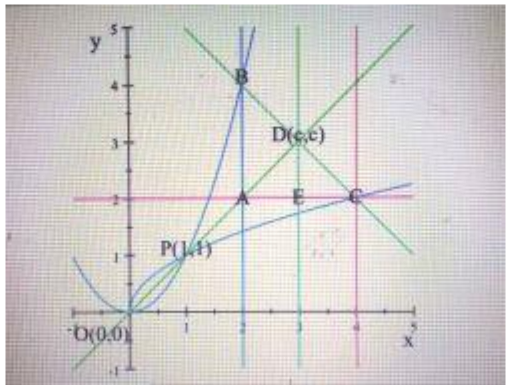

Figure-1

As plotted above in the figure, curves $y = x^{\alpha}$ and $y = x^{\frac{1}{\alpha}}$ are symmetric about line $y = x$, pints B and C respectively lies at the intersection of the curves $y = x^{\frac{1}{\alpha}}$ and $y = x^{\alpha}$, and the line perpendicular to line $y = x$ and passing through point $D(c, c)$. Thus in ΔABC, AE=EC=d (say). Therefore x-coordinate of point C is c+d,and x-coordinate of point B is c-d.Since y=x^{α}, and line passing through ponits B, C and D(c,c) i.e. $y = -x + 2c$ intersects each other at point C, therfore at C we must have $ax^{\alpha} = -bx + 2$.

$$ax^{\alpha} + bx - 2c = 0. \tag{2.7}$$

Since x-coordinate of point C is $c + d$, therefore at C we must have

$$(c + d)^{\alpha} + c + d - 2c = 0,$$

$$(c + d)^{\alpha} = c - d. \tag{2.8}$$

Similarly at point B, x-coordinate of point B is $c - d$, therefore at B we must have

$$(c - d)^{\frac{1}{\alpha}} = c + d. \tag{2.9}$$

Thus to find root of equation (2.7) we need to find d in terms of c and α, for this rewrite equation (2.8) as

$$(1 + f)^{\alpha} = c^{1-\alpha}(1 - f) \tag{2.10}$$

Where $f = (d/c)$. To find f, rewrite equation (2.10) as

$$\frac{(1+f)^{\alpha}}{(1-f)} = c^{1-\alpha}.$$

This gives

$$\frac{1}{2\pi i(1-f)} \oint_{|z|=1} \frac{(1+z)^{\alpha}}{z^{\alpha+1}} dz + \frac{f}{2\pi i(1-f)} \oint_{|z|=1} \frac{(1+z)^{\alpha}}{(1-fz)z^{\alpha}} dz = c^{1-\alpha}. \tag{2.11}$$

Solve equation (2.11) as $z = -1$ *and* $z = 0$ are branch points it gives

$$f = \frac{c^{2-\alpha}}{2^{\alpha}\frac{\sin\pi\alpha}{\pi} + c^{1-\alpha}} - c^{\alpha}.$$

Therefore

$$c + d = c + fc = c + \frac{c^{2-\alpha}}{2^{\alpha}\frac{\sin\pi\alpha}{\pi} + c^{1-\alpha}} - c^{\alpha},$$

is a root of equation (2.7). And

$$c - d = c - fc = c - \frac{c^{2-\alpha}}{2^{\alpha}\frac{\sin\pi\alpha}{\pi} + c^{1-\alpha}} + c^{\alpha},$$

is a root of equation (2.9)

Conclusion: *Thus with the help of result obtained in theorem 1, Theorem2 and Theorem 3, we can find root of equations of the form*

$$ax^{\alpha} + bx^{\beta} + c = 0.$$

Where a, b, c, α, β are real numbers and $\alpha > \beta$.

Remarks 3. *Below are the references which have been consulted for motivation and some intial idea, but have not been used anywhere in this article.*

Remark 4. *There is no involvement of any form of animal life during this research, and there is no conflict of interest with any one. There is no institutional funding received for this research.*

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References

[1] Abel, Niels Henrik (1881) [1824], "Mémoire sur les équations algébriques, ou l'on démontre l'impossibilité de la résolution de l'équation générale du cinquième degré" (PDF), in Sylow, Ludwig; Lie, Sophus (eds.), Œuvres Complètes de Niels Henrik Abel (in French), vol. I (2nd ed.), Grøndahl & Søn, pp. 28--33.

[2] Abel, Niels Henrik (1881) [1826], "Démonstration de l'impossibilité de la résolution algébrique des équations générales qui passent le quatrième degré" (PDF), in Sylow, Ludwig; Lie, Sophus (eds.), Œuvres Complètes de Niels Henrik Abel (in French), vol. I (2nd ed.), Grøndahl & Søn, pp. 66--87.

[3] Ayoub, Raymond G. (1980), "Paolo Ruffini's Contributions to the Quintic", Archive for History of Exact Sciences, 22 (3): 253--277, doi:10.1007/BF00357046.

[4] Bradley, Michael. The Birth of Mathematics: Ancient Times to 1300, p. 86 (Infobase Publishing 2006).

[5] https://en.wikipedia.org/wiki/Quadratic_formula.

[6] Ruffini, Paolo (1813). Riflessioni intorno alla soluzione delle equazioni algebraiche generali opuscolo del cav. dott. Paolo Ruffini.

[7] Sterling, Mary Jane (2010), Algebra I For Dummies, Wiley Publishing, p. 219, ISBN 978-0-470-55964-2.

[8] Struik, D. J.; Stevin, Simon (1958), The Principal Works of Simon Stevin, Mathematics (PDF), vol. II--B, C. V. Swets & Zeitlinger, p. 470.

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