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Multi (n) Qubit Quantum Teleportation Framework Using Qiskit

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Abstract

Quantum teleportation allows us to break down a quantum state in one location and reconstruct it another location. This paper proposes a general framework for multi qubit quantum teleportation with more complex entanglement. Simultaneously, we verify these results by coding these quantum circuits using the qiskit library in python.

Introduction

Quantum teleportation is of immense importance in quantum computers and one of the fundamental ideas in quantum computing. We start with a basic overview of quantum teleportation using EPR pairs [1] as outlined in the original Quantum Teleportation paper [2] along with its required quantum circuit. We then move onto two qubit teleportation where we give the quantum circuit and all the mathematical steps (Appendix A) to show that this two-qubit protocol works. We then prove this result with the help of the qiskit library in python and a quantum simulator. Finally, we compare the quantum circuits for one and two qubit teleportation protocols to develop a general framework for n qubit quantum teleportation. We verify this generalized teleportation protocol with the qiskit library in python

Quantum Teleportation.

The basic idea and setup to transport an arbitrary qubit is as follows: -

- Let there be two people - Alice and Bob. They create an entangled state in the maximally entangled Bell state B_{00} and each of them keep one qubit from it. Let Alice's qubit be ‘A’ and Bob’s be ‘B’

- Now if Alice and Bob are at different locations and each still have their part of the Bell state, Alice can send the quantum state of another separate system (Say Ψ) to Bob using their shared entangled state.

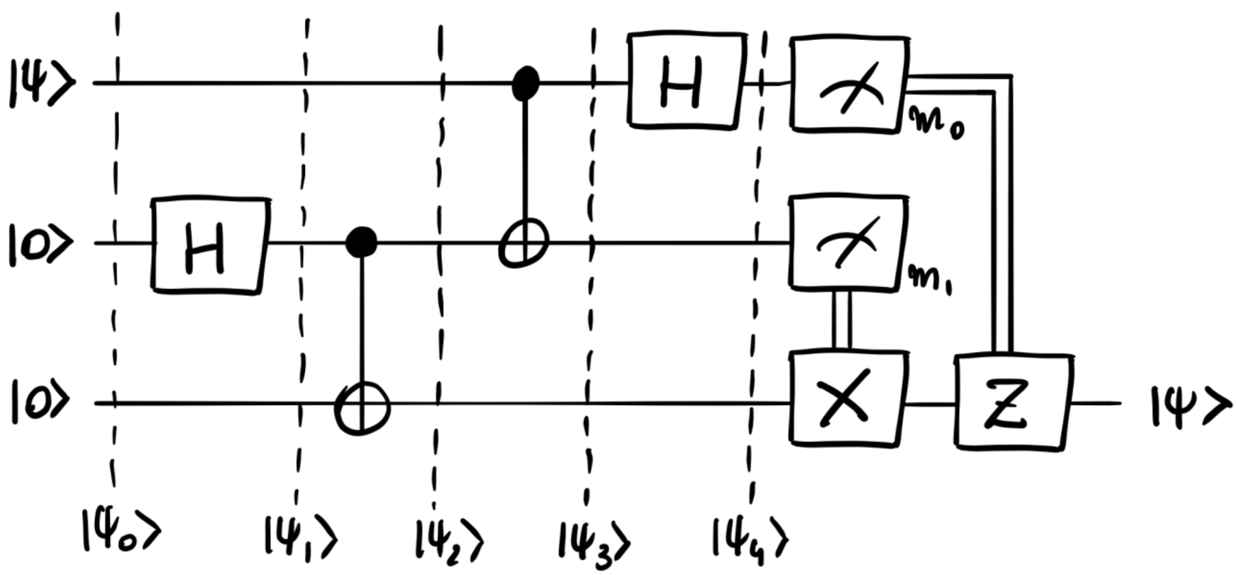
- To do this, Alice would have to perform some operations on Ψ and A (her part of the Bell State). Note that this will cause Bobs half of the Bell state to change as well since they are entangled

- Now Alice would make a measurement on the result of her operations which would give her two classical bits which she would send to Bob through classical methods

- On the basis of the classical bits Bob receives, he would have to perform some unitary transformations on his half of the Bell state, B

- The result of these transformations would be the state Alice wanted to send and the state would have been successfully transported

Mathematically this can be shows as follows: -



This is the quantum circuit for a quantum teleportation protocol where |Ψ0⟩, |Ψ1⟩, |Ψ2⟩, etc are the quantum state of the 3-qubit system at that point in time. Let the arbitrary qubit |Ψ⟩ = a|0⟩ + b|1

Note: subscripts on gates indicate qubit at what index the gates are acting on. Indexing begins from 0

$|\Psi_0\rangle = |0\rangle|0\rangle = |00\rangle$

$|\Psi_1\rangle = H_1\,|00\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle)\,|0\rangle = 1/\sqrt{2}\,(|00\rangle + |10\rangle)$

$|\Psi_2\rangle = CNOT_{1,2}\; 1/\sqrt{2}\,(|00\rangle + |10\rangle) = 1/\sqrt{2}\,(\,|00\rangle + |11\rangle) = B00$

At this point the maximally entangled bell State has been prepared. Half of the bell state is kept with Alice while the other half is with Bob. The following protocol for teleportation will work regardless of the distance between Alice and Bob.

$|\Psi_3\rangle = CNOT_{0,1}\,(|\Psi\rangle\, B00) = CNOT_{0,1}\,[\,(1/\sqrt{2}\,(a|0\rangle + b|1\rangle)\,(|00\rangle + |11\rangle)]$

$= CNOT_{0,1}\,[\,1/\sqrt{2}\,(a|000\rangle + a|011\rangle + b|100\rangle + b|111\rangle)]$

$= 1/\sqrt{2}\,(a|000\rangle + a|011\rangle + b|110\rangle + b|101\rangle)$

$|\Psi_4\rangle = H_0\,[\,1/\sqrt{2}\,(a|000\rangle + a|011\rangle + b|110\rangle + b|101\rangle)]$

$= 1/\sqrt{2}[\,a/\sqrt{2}\,(|0\rangle + |1\rangle)\,|00\rangle + a/\sqrt{2}\,(|0\rangle + |1\rangle)\,|11\rangle + b/\sqrt{2}\,(|0\rangle - |1\rangle)\,|10\rangle + b/\sqrt{2}\,(|0\rangle - |1\rangle)\,|01\rangle]$

$= \frac{1}{2}$ [a|000⟩ + a|100⟩ + a|011⟩ + a|111⟩ + b|010⟩ - b|110⟩ + b|001⟩ - b|010⟩]

$= \frac{1}{2}$ [|00⟩ (a|0⟩ + b|1⟩) + |10⟩ (a|0⟩ - b|1⟩) + |01⟩ (a|1⟩ + b|0⟩) + |11⟩ (a|1⟩ - b|0⟩)]

We can see that this state $|\Psi_4\rangle$ is entangled such that if first two qubits (Alice's) are measured to be |10⟩ then the third qubit (Bob's) will be in the superposition state a|0⟩ - b|1⟩. Now depending on the result of the measurement, some unitary transformations need to performed on Bob's qubit to convert it to the state |Ψ⟩ = a|0⟩ + b|1⟩ which is being transported. This can be seen from the below table

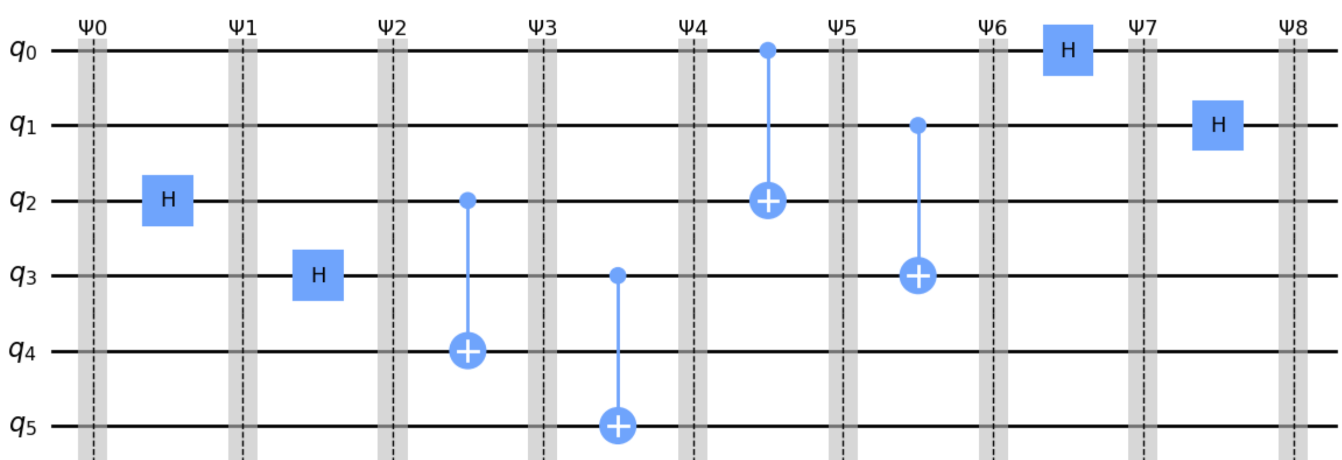
Measurement on Ψ and 'A'	Probability	Resultant state of 'B'	Transformations
00	1/4	a|0⟩ + b|1⟩	I
01	1/4	a|1⟩ + b|0⟩	X
10	1/4	a|0⟩ - b|1⟩	Z
11	1/4	a|1⟩ - b|0⟩	XZ

Once these transformations are done, the state will have been successfully transported.

2 Qubit Teleportation

First, I developed a quantum circuit to transport 2 qubits using an entangled state of 4 qubits with perfect correlation between first two and last two qubits. I.e., If the first two qubits are in the state 'xy' then the last two qubits also have to be in the state 'xy', where x = 0 or 1 and y = 0 or 1

Let the 2 arbitrary qubits to be transported be represented by p|0⟩ + q|1⟩ and r|0⟩ + s|1⟩. Then their joint state is given by |Ψ⟩ = (p|0⟩ + q|1⟩) (r|0⟩ + s|1⟩) = pr|00⟩ + ps|01⟩ + qr|10⟩ + qr|11⟩. Let the normalized form of this state be |Ψ⟩ = a|00⟩ + b|01⟩ + c|10⟩ + d|11⟩ such that $a^2 + b^2 + c^2 + d^2 = 1$. Then assuming this state is stored in the first two qubits, the quantum circuit (made with qiskit) for teleporting a 2-qubit state is as follows



$|\Psi_0\rangle$ = (a|00⟩ + b|01⟩ + c|10⟩ + d|11⟩) |0⟩|0⟩|0⟩|0⟩ = (a|00⟩ + b|01⟩ + c|10⟩ + d|11⟩) |0000⟩

$|\Psi_8\rangle = (H_1 (H_0 (CNOT_{1,3} (CNOT_{0,2} (CNOT_{3,5} (CNOT_{2,4} (H_3 (H_2 (|\Psi_0\rangle)))))))))$

= ¼ (a (|00⟩ + |01⟩ + |10⟩ + |11⟩) |0000⟩ + a (|00⟩ + |01⟩ + |10⟩ + |11⟩) |0101⟩ + a(|00⟩ + |01⟩ + |10⟩ + |11⟩) |1010⟩ + a(|00⟩ + |01⟩ + |10⟩ + |11⟩) |1111⟩ + b(|00⟩ - |01⟩ + |10⟩ - |11⟩) |0100⟩ +b(|00⟩ - |01⟩ + |10⟩ - |11⟩) |0001⟩ + b(|00⟩ - |01⟩ + |10⟩ - |11⟩) |1110⟩ + b(|00⟩ - |01⟩ + |10⟩ - |11⟩) |1011⟩ + c(|00⟩ + |01⟩ - |10⟩ - |11⟩) |1000⟩ + c(|00⟩ + |01⟩ - |10⟩ - |11⟩)|1101⟩ + c(|00⟩ + |01⟩ - |10⟩ - |11⟩)|0010⟩ + c (|00⟩ + |01⟩ - |10⟩ - |11⟩) |0111⟩ + d (|00⟩ - |01⟩ - |10⟩ + |11⟩) |1100⟩ + d (|00⟩ - |01⟩ - |10⟩ + |11⟩) |1001⟩ + d (|00⟩ - |01⟩ - |10⟩ + |11⟩) |0110⟩ + d (|00⟩ - |01⟩ - |10⟩ + |11⟩) |0011⟩

= ¼ (a|000000⟩ + a|010000⟩ + a|100000⟩ + a|110000⟩ + a|000101⟩ + a|010101⟩ + a|100101⟩ + a|110101⟩ + a|001010⟩ + a|011010⟩ + a|101010⟩ + a|111010⟩ + a|001111⟩ + a|011111⟩ + a|101111⟩ + a|111111⟩ + b|000100⟩ - b|010100⟩ + b|100100⟩ - b|110100⟩ + b|000001⟩ - b|010001⟩ + b|100001⟩ - b|110001⟩ + b|001110⟩ - b|011110⟩ + b|101110⟩ - b|111110⟩ + b|001011⟩ - b|011011⟩ + b|101011⟩ - b|111011⟩ + c|001000⟩ + c|011000⟩ - c|101000⟩ - c|111000⟩ + c|001101⟩ + c|011101⟩ - c|101101⟩ - c|111101⟩ + c|000010⟩ + c|010010⟩ - c|100010⟩ - c|110010⟩ + c|000111⟩ + c|010111⟩ - c|100111⟩ - c|110111⟩ + d|001100⟩ - d|011100⟩ - d|101100⟩ + d|111100⟩ + d|001001⟩ - d|011001⟩ - d|101001⟩ + d|111001⟩ + d|000110⟩ - d|010110⟩ - d|100110⟩ + d|110110⟩ + d|000011⟩ - d|010011⟩ - d|100011⟩ + d|110011⟩)

= ¼ [|0000⟩ (a|00⟩ + b|01⟩ + c|10⟩ + d|11⟩) + |0001⟩ (a|01⟩ + b|00⟩ + c|11⟩ + d|10⟩) +
|0010⟩ (a|10⟩ + b|11⟩ + c|00⟩ + d|01⟩) + |0011⟩ (a|11⟩ + b|10⟩ + c|01⟩ + d|00⟩) +
|0100⟩ (a|00⟩ - b|01⟩ + c|10⟩ - d|11⟩) + |1000⟩ (a|00⟩ + b|01⟩ - c|10⟩ - d|11⟩) +
|1100⟩ (a|00⟩ - b|01⟩ - c|10⟩ + d|11⟩) + |0101⟩ (a|01⟩ - b|00⟩ + c|11⟩ - d|10⟩) +
|1001⟩ (a|01⟩ + b|00⟩ - c|11⟩ - d|10⟩) + |1101⟩ (a|01⟩ - b|00⟩ - c|11⟩ + d|10⟩) +
|0110⟩ (a|10⟩ - b|11⟩ + c|00⟩ - d|01⟩) + |1010⟩ (a|10⟩ + b|11⟩ - c|00⟩ - d|01⟩) +
|1110⟩ (a|10⟩ - b|11⟩ - c|00⟩ + d|01⟩) + |0111⟩ (a|11⟩ - b|10⟩ + c|01⟩ - d|00⟩) +
|1011⟩ (a|11⟩ + b|10⟩ - c|01⟩ - d|00⟩) + |1111⟩ (a|11⟩ - b|10⟩ - c|01⟩ + d|00⟩)

See appendix A for step-by-step calculations

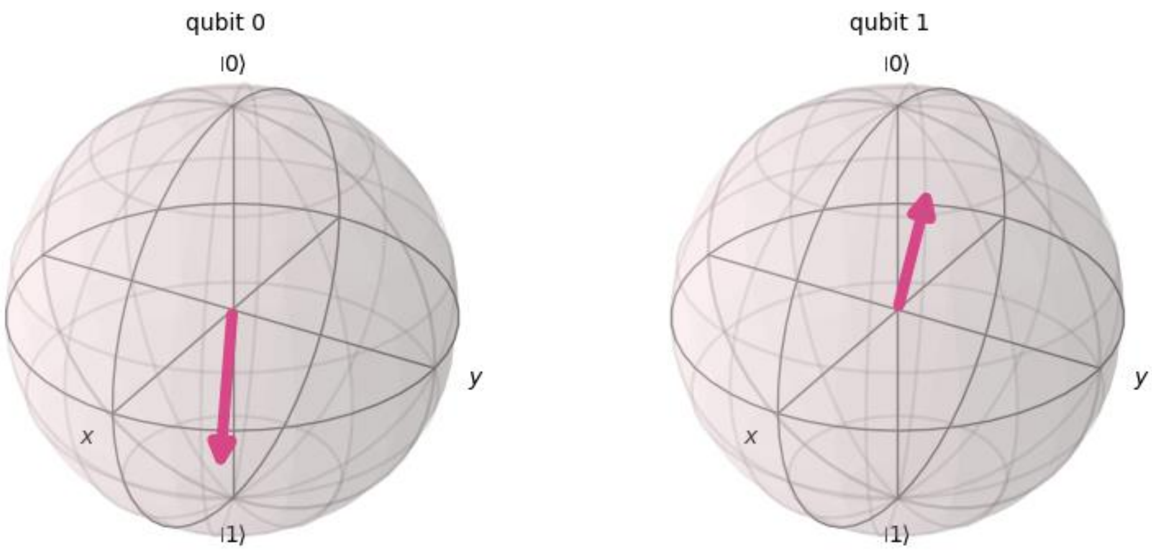
As it can be seen from this final entangled state, when Alice's four qubits are in the state |0000⟩, Bob's two qubits would be in the state a|00⟩ + b|01⟩ + c|10⟩ + d|11⟩ and so on. In some of the cases, like in one qubit teleportation, unitary transformations need to be made on Bob's qubits. It can also be noted that all measurements/result states in the final entangled state have an equal probability of $(¼)^2$ = 1/16. Once again, in all the cases, the only gates needed for the transformations are the Pauli X and Z gate. However, the transformations are slightly more complicated because there are two possible qubits that Bob has which the gates could be applied to. The gates for all possibilities are outlines in the table below

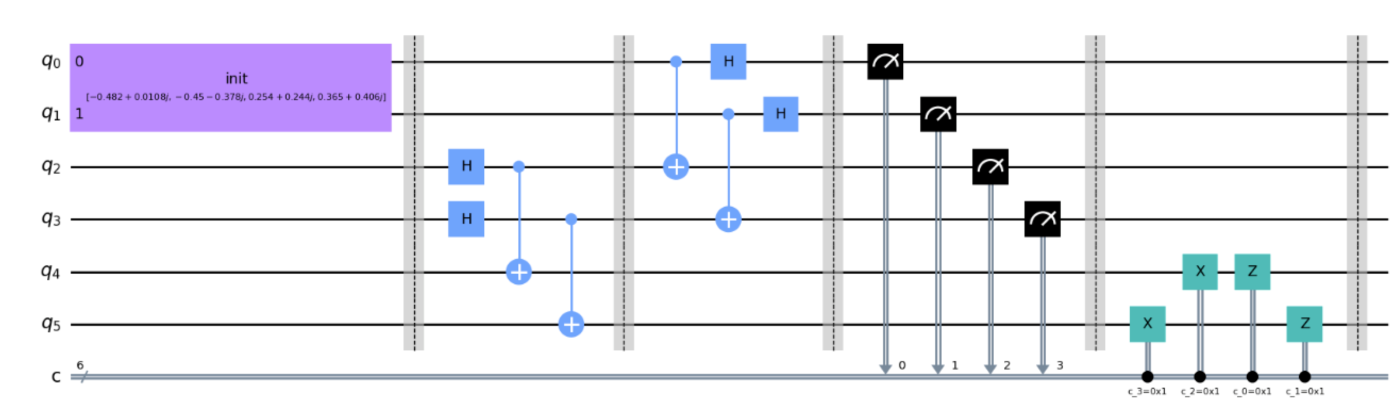
Measurement on qubits with Alice	Probability	Resultant state of Bob's qubits (B)	Transformations
$\|0000\rangle$	1/16	$a\|00\rangle + b\|01\rangle + c\|10\rangle + d\|11\rangle$	I
$\|0001\rangle$	1/16	$a\|01\rangle + b\|00\rangle + c\|11\rangle + d\|10\rangle$	X_5
$\|0010\rangle$	1/16	$a\|10\rangle + b\|11\rangle + c\|00\rangle + d\|01\rangle$	X_4
$\|0011\rangle$	1/16	$a\|11\rangle + b\|10\rangle + c\|01\rangle + d\|00\rangle$	$X_4 X_5$
$\|0100\rangle$	1/16	$a\|00\rangle - b\|01\rangle + c\|10\rangle - d\|11\rangle$	Z_5
$\|1000\rangle$	1/16	$a\|00\rangle + b\|01\rangle - c\|10\rangle - d\|11\rangle$	Z_4
$\|1100\rangle$	1/16	$a\|00\rangle - b\|01\rangle - c\|10\rangle + d\|11\rangle$	$Z_4 Z_5$
$\|0101\rangle$	1/16	$a\|01\rangle - b\|00\rangle + c\|11\rangle - d\|10\rangle$	$Z_5 X_5$
$\|1001\rangle$	1/16	$a\|01\rangle + b\|00\rangle - c\|11\rangle - d\|10\rangle$	$Z_4 X_5$
$\|1101\rangle$	1/16	$a\|01\rangle - b\|00\rangle - c\|11\rangle + d\|10\rangle$	$Z_4 Z_5 X_5$
$\|0110\rangle$	1/16	$a\|10\rangle - b\|11\rangle + c\|00\rangle - d\|01\rangle$	$Z_5 X_4$
$\|1010\rangle$	1/16	$a\|10\rangle + b\|11\rangle - c\|00\rangle - d\|01\rangle$	$Z_4 X_4$
$\|1110\rangle$	1/16	$a\|10\rangle - b\|11\rangle - c\|00\rangle + d\|01\rangle$	$Z_4 Z_5 X_4$

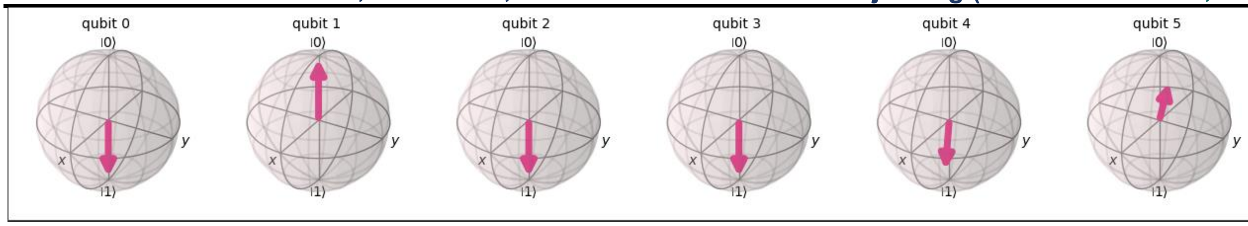
|0111⟩	1/16	a|11⟩ - b|10⟩ + c|01⟩ - d|00⟩	$Z_5 X_4 X_5$
|1011⟩	1/16	a|11⟩ + b|10⟩ - c|01⟩ - d|00⟩	$Z_4 X_4 X_5$
|1111⟩	1/16	a|11⟩ - b|10⟩ - c|01⟩ + d|00⟩	$Z_4 Z_5 X_4 X_5$

Once these transformations are performed, 2 qubits will have been successfully transported from Alice to Bob.

These calculations can be rechecked using the qiskit library in python to build a quantum circuit and run it through a perfect quantum simulator. The code for building the circuit is shown in Appendix B and the output we get from it is as follows







The first image shows the initial state of the first 2 qubits while the second image shows the quantum circuit used. The third image shows the state of all 6 qubits used at the end of the teleportation protocol. From the initial and final Bloch sphere representation, we can clearly see that the qubits initialized to the first 2 qubits were transported to the last 2 qubits while all the other qubits collapsed to a value of 0 or 1 since they were measured. Note that however many times you run the code, the final two qubits will always be in the state psi that was being transported. But the other 4 qubits could collapse to any of the 16 combinations of 0 and 1's as indicated by the calculations.

Discussion and Results

On comparing the two circuits for single and two qubit teleportation we can see a pattern that we can use to create a general framework for 'n' qubit teleportation. Firstly, we see that the circuit requires 3n qubits if n qubits are to be transported. These can be separated into 3 groups of 'n' qubits called Ψ, 'A' and 'B' in that order. Ψ is the state to be transported and all qubits in 'A' and 'B' are in state $|0\rangle$.

The teleportation protocol can be generalized in the following way -

1) A Hadamard gate is performed on every qubit in the group 'A'.

2) A CNOT gate is performed from each qubit in 'A' to its corresponding qubit in 'B'. i.e., CNOT gate from 1st qubit of 'A' to 1st qubit of 'B', CNOT gate from 2nd qubit of 'A' to 2nd qubit of 'B', etc, where the qubit in A is the control. At this point the maximally entangled state has been prepared.

3) A CNOT gate is performed from each qubit in Ψ to its corresponding qubit in 'A'

4) A Hadamard gate is performed on every qubit in Ψ

5) Each qubit in Ψ and 'A' is measured to give 2n classical bits, leaving 'n' qubits of B

6) Depending on the values of the 2n classical bits measured, some unitary transformations are performed. Specifically, an X gate is applied on the i^{th} qubit (of the ones remaining) if the $(n + i)^{th}$ classical bit has value 1. Similarly, a Z gate is applied on the i^{th} qubit (of the ones remaining) if the i^{th} classical bit has value 1

Once again, we can check that this method is correct by running a quantum simulation using the qiskit library in python. The following code does the same

```
n = int(input('How many qubits how would you like to teleport:'))

psi_n = random_state(n)
array_to_latex(psi_n, pretext="|\\psi\\rangle =")
plot_bloch_multivector(psi_n)
```

```
init_gate = Initialize(psi_n)
init_gate.label = "init"

qc = QuantumCircuit(3*n,3*n)

qc.append(init_gate, [i for i in range(n)])
qc.barrier()

for i in range(n):
  qc.h(n+i)

for i in range(n):
  qc.cx(n+i, (2*n)+i)
qc.barrier()

for i in range(n):
  qc.cx(i, n+i)

for i in range(n):
  qc.h(i)
qc.barrier()

for i in range(2*n):
  qc.measure(i, i)
qc.barrier()

for i in range(n):
  qc.x((2*n + i)).c_if(n+i, 1)

for i in range(n):
  qc.z((2*n + i)).c_if(i, 1)

qc.draw()
```

```
backend = BasicAer.get_backend('statevector_simulator')
result = execute(qc, backend).result()
out_vector = result.get_statevector()
plot_bloch_multivector(out_vector, figsize=(6,6))
```

Therefore, we have successfully developed a framework for teleporting multiple qubits by using an entangled state with higher degrees of freedom. This can also be thought of as transporting each qubit individually

This work can be taken forward by attempting to transport multiple qubit experimentally over long distances in a similar manner to that achieved for a single qubit [5].

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Appendix A

Full calculations for 2 qubit teleportation

$|\Psi_0\rangle = (a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle)\ |0\rangle|0\rangle|0\rangle|0\rangle = (a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle)\ |0000\rangle$

Note: We will be ignoring the state to be teleported for the first 4 steps as no gates act on it in this time.

$|\Psi_1\rangle = H_2\ (|0000\rangle) = 1/\sqrt{2}(|0\rangle + |1\rangle)\ (|000\rangle) = 1/\sqrt{2}\ (|0000\rangle + |1000\rangle)$

$|\Psi_2\rangle = H_3\ (1/\sqrt{2}\ (|0000\rangle + |1000\rangle)) = ½\ (|0\rangle\ (|0\rangle + |1\rangle)\ |00\rangle + |1\rangle\ (|0\rangle + |1\rangle)\ |00\rangle)$

$= ½\ (|0000\rangle + |0100\rangle + |1000\rangle + |1100\rangle)$

$|\Psi_3\rangle = CNOT_{2,4}\ (½\ (|0000\rangle + |0100\rangle + |1000\rangle + |1100\rangle)) = ½\ (|0000\rangle + |0100\rangle + |1010\rangle + |1110\rangle)$

$|\Psi_4\rangle = CNOT_{3,5}\ (½\ (|0000\rangle + |0100\rangle + |1010\rangle + |1110\rangle)) = ½\ (|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle)$

Note: The entangled state was being created upto this point. The teleportation protocol begins from here

$|\Psi_4\rangle = ½\ (a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle)\ (|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle)$

$= ½\ (a|000000\rangle + a|000101\rangle + a|001010\rangle + a|001111\rangle + b|010000\rangle + b|010101\rangle + b|011010\rangle +$

$b|011111\rangle + c|100000\rangle + c|100101\rangle + c|101010\rangle + c|101111\rangle + d|110000\rangle + d|110101\rangle+$

$d|111010\rangle + d|111111\rangle$

$|\Psi_5\rangle = CNOT_{0,2}\,(|\Psi_5\rangle) = \frac{1}{2}\,(a|000000\rangle + a|000101\rangle + a|001010\rangle + a|001111\rangle + b|010000\rangle + b|010101\rangle +$

$b|011010\rangle + b|011111\rangle + c|101000\rangle + c|101101\rangle + c|100010\rangle + c|100111\rangle + d|111000\rangle +$
$d|111101\rangle + d|110010\rangle + d|110111\rangle$

$|\Psi_6\rangle = CNOT_{1,3}\,(|\Psi_6\rangle) = \frac{1}{2}\,(a|000000\rangle + a|000101\rangle + a|001010\rangle + a|001111\rangle + b|010100\rangle + b|010001\rangle +$

$b|011110\rangle + b|011011\rangle + c|101000\rangle + c|101101\rangle + c|100010\rangle + c|100111\rangle + d|111100\rangle +$
$d|111001\rangle + d|110110\rangle + d|110011\rangle$

$|\Psi_7\rangle = H_0\,(|\Psi_7\rangle) = \frac{1}{2}\,(a/\sqrt{2}(|0\rangle + |1\rangle)\,|00000\rangle + a/\sqrt{2}(|0\rangle + |1\rangle)\,|00101\rangle + a/\sqrt{2}(|0\rangle + |1\rangle)\,|01010\rangle +$

$a/\sqrt{2}(|0\rangle + |1\rangle)\,|01111\rangle + b/\sqrt{2}(|0\rangle + |1\rangle)\,|10100\rangle + b/\sqrt{2}(|0\rangle + |1\rangle)\,|10001\rangle +$
$b/\sqrt{2}(|0\rangle + |1\rangle)\,|11110\rangle + b/\sqrt{2}(|0\rangle + |1\rangle)\,|11011\rangle + c/\sqrt{2}(|0\rangle - |1\rangle)\,|01000\rangle + c/\sqrt{2}(|0\rangle$
$- |1\rangle)\,|01101\rangle + c/\sqrt{2}(|0\rangle - |1\rangle)\,|00010\rangle + c/\sqrt{2}(|0\rangle - |1\rangle)\,|00111\rangle + d/\sqrt{2}(|0\rangle - |1\rangle)\,|11100\rangle$
$+ d/\sqrt{2}(|0\rangle - |1\rangle)\,|11001\rangle + d/\sqrt{2}(|0\rangle - |1\rangle)\,|10110\rangle + d/\sqrt{2}(|0\rangle - |1\rangle)\,|10011\rangle$

$|\Psi_8\rangle = H_1\,(|\Psi_8\rangle) = \frac{1}{4}\,(a\,(|0\rangle + |1\rangle)\,(|0\rangle + |1\rangle)\,|0000\rangle + a\,(|0\rangle + |1\rangle)\,(|0\rangle + |1\rangle)\,|0101\rangle +$
$a\,(|0\rangle + |1\rangle)\,(|0\rangle + |1\rangle)\,|1010\rangle + a\,(|0\rangle + |1\rangle)\,(|0\rangle + |1\rangle)\,|1111\rangle + b\,(|0\rangle + |1\rangle)\,(|0\rangle - |1\rangle)\,|0100\rangle +$
$b\,(|0\rangle + |1\rangle)\,(|0\rangle - |1\rangle)\,|0001\rangle + b\,(|0\rangle + |1\rangle)\,(|0\rangle - |1\rangle)\,|1110\rangle + b\,(|0\rangle + |1\rangle)\,(|0\rangle - |1\rangle)\,|1011\rangle +$
$c\,(|0\rangle - |1\rangle)\,(|0\rangle + |1\rangle)\,|1000\rangle + c\,(|0\rangle - |1\rangle)\,(|0\rangle + |1\rangle)\,|1101\rangle + c\,(|0\rangle - |1\rangle)\,(|0\rangle + |1\rangle)\,|0010\rangle +$
$c\,(|0\rangle - |1\rangle)\,(|0\rangle + |1\rangle)\,|0111\rangle + d\,(|0\rangle - |1\rangle)\,(|0\rangle - |1\rangle)\,|1100\rangle + d\,(|0\rangle - |1\rangle)\,(|0\rangle - |1\rangle)\,|1001\rangle +$
$d\,(|0\rangle - |1\rangle)\,(|0\rangle - |1\rangle)\,|0110\rangle + d\,(|0\rangle - |1\rangle)\,(|0\rangle - |1\rangle)\,|0011\rangle)$

$= \frac{1}{4}\,(a\,(|00\rangle + |01\rangle + |10\rangle + |11\rangle)\,|0000\rangle + a\,(|00\rangle + |01\rangle + |10\rangle + |11\rangle)\,|0101\rangle +$
$a(|00\rangle + |01\rangle + |10\rangle + |11\rangle)\,|1010\rangle + a(|00\rangle + |01\rangle + |10\rangle + |11\rangle)\,|1111\rangle + b(|00\rangle - |01\rangle + |10\rangle - |11\rangle)\,|0100\rangle + b($
$|00\rangle - |01\rangle + |10\rangle - |11\rangle)\,|0001\rangle + b(|00\rangle - |01\rangle + |10\rangle - |11\rangle)\,|1110\rangle + b(|00\rangle - |01\rangle + |10\rangle - |11\rangle)\,|1011\rangle + c(|00\rangle$
$+ |01\rangle - |10\rangle - |11\rangle)\,|1000\rangle + c(|00\rangle + |01\rangle - |10\rangle - |11\rangle)|1101\rangle + c(|00\rangle + |01\rangle - |10\rangle - |11\rangle)|0010\rangle +$
$c\,(|00\rangle + |01\rangle - |10\rangle - |11\rangle)\,|0111\rangle + d\,|00\rangle - |01\rangle - |10\rangle + |11\rangle)\,|1100\rangle + d\,(|00\rangle - |01\rangle - |10\rangle + |11\rangle)\,|1001\rangle +$
$d\,(|00\rangle - |01\rangle - |10\rangle + |11\rangle)\,|0110\rangle + d\,(|00\rangle - |01\rangle - |10\rangle + |11\rangle)\,|0011\rangle$

$= \frac{1}{4}\,(a|000000\rangle + a|010000\rangle + a|100000\rangle + a|110000\rangle + a|000101\rangle + a|010101\rangle + a|100101\rangle + a|110101\rangle$
$+ a|001010\rangle + a|011010\rangle + a|101010\rangle + a|111010\rangle + a|001111\rangle + a|011111\rangle + a|101111\rangle + a|111111\rangle + b|000100\rangle$
$- b|010100\rangle + b|100100\rangle - b|110100\rangle + b|000001\rangle - b|010001\rangle + b|100001\rangle - b|110001\rangle + b|001110\rangle - b|011110\rangle +$
$b|101110\rangle - b|111110\rangle + b|001011\rangle - b|011011\rangle + b|101011\rangle - b|111011\rangle + c|001000\rangle + c|011000\rangle - c|101000\rangle -$
$c|111000\rangle + c|001101\rangle + c|011101\rangle - c|101101\rangle - c|111101\rangle + c|000010\rangle + c|010010\rangle - c|100010\rangle - c|110010\rangle +$
$c|000111\rangle + c|010111\rangle - c|100111\rangle - c|110111\rangle + d|001100\rangle - d|011100\rangle - d|101100\rangle + d|111100\rangle + d|001001\rangle -$
$d|011001\rangle - d|101001\rangle + d|111001\rangle + d|000110\rangle - d|010110\rangle - d|100110\rangle + d|110110\rangle + d|000011\rangle - d|010011\rangle -$
$d|100011\rangle + d|110011\rangle)$

$= \frac{1}{4}\,[\,|0000\rangle\,(a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle) + |0001\rangle\,(a|01\rangle + b|00\rangle + c|11\rangle + d|10\rangle) +$
$|0010\rangle\,(a|10\rangle + b|11\rangle + c|00\rangle + d|01\rangle) + |0011\rangle\,(a|11\rangle + b|10\rangle + c|01\rangle + d|00\rangle) +$
$|0100\rangle\,(a|00\rangle - b|01\rangle + c|10\rangle - d|11\rangle) + |1000\rangle\,(a|00\rangle + b|01\rangle - c|10\rangle - d|11\rangle) +$
$|1100\rangle\,(a|00\rangle - b|01\rangle - c|10\rangle + d|11\rangle) + |0101\rangle\,(a|01\rangle - b|00\rangle + c|11\rangle - d|10\rangle) +$
$|1001\rangle\,(a|01\rangle + b|00\rangle - c|11\rangle - d|10\rangle) + |1101\rangle\,(a|01\rangle - b|00\rangle - c|11\rangle + d|10\rangle) +$
$|0110\rangle\,(a|10\rangle - b|11\rangle + c|00\rangle - d|01\rangle) + |1010\rangle\,(a|10\rangle + b|11\rangle - c|00\rangle - d|01\rangle) +$
$|1110\rangle\,(a|10\rangle - b|11\rangle - c|00\rangle + d|01\rangle) + |0111\rangle\,(a|11\rangle - b|10\rangle + c|01\rangle - d|00\rangle) +$
$|1011\rangle\,(a|11\rangle + b|10\rangle - c|01\rangle - d|00\rangle) + |1111\rangle\,(a|11\rangle - b|10\rangle - c|01\rangle + d|00\rangle)$

Appendix B

Code for 2 qubit quantum teleportation circuit with qiskit

```
psi_2 = random_state(2)
array_to_latex(psi_2, pretext="|\\psi\\rangle =")
plot_bloch_multivector(psi_2)
```

```
init_gate = Initialize(psi_2)
init_gate.label = "init"
qc = QuantumCircuit(6,6)

qc.append(init_gate, [0,1])
qc.barrier()

qc.h(2)
qc.h(3)
qc.cx(2,4)
qc.cx(3,5)
qc.barrier()

qc.cx(0, 2)
qc.cx(1, 3)
qc.h(0)
qc.h(1)

qc.barrier()
qc.measure(0,0)
qc.measure(1,1)
qc.measure(2,2)
qc.measure(3,3)
qc.barrier()

qc.x(5).c_if(3,1)
qc.x(4).c_if(2,1)
qc.z(4).c_if(0,1)
qc.z(5).c_if(1,1)

qc.barrier()
qc.draw()
```

```
backend = BasicAer.get_backend('statevector_simulator')
result = execute(qc, backend).result()
out_vector = result.get_statevector()
plot_bloch_multivector(out_vector)
```

Appendix C

Imports required for the quantum circuits

```
!pip install qiskit
!pip install git+https://github.com/qiskit-community/qiskit-textbook.git#subdirectory=qiskit-textbook-src
import qiskit
import numpy as np
from qiskit.extensions import Initialize
from qiskit import QuantumCircuit, QuantumRegister
from qiskit import ClassicalRegister, execute, BasicAer, IBMQ
from qiskit_textbook.tools import random_state, array_to_latex
from qiskit.visualization import plot_histogram, plot_bloch_multivector
%matplotlib inline
```